Estimates for entropy numbers of embedding operators of function spaces on sets with tree-like structure: some limiting cases

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1 Introduction

In [37] order estimates for entropy numbers of the embedding operator of a weighted Sobolev space on a John domain into a weighted Lebesgue space were obtained, as well as estimates for entropy numbers of a two-weighted summation operator on a tree. Here we consider some critical cases.

Recall the definition of entropy numbers (see, e.g., [6, 9, 32]).

Definition 1. *Let X, Y be normed spaces, and let $T : X \to Y$ be a linear continuous operator. Entropy numbers of T are defined by*

$$e_k(T) = \inf\left\{\varepsilon > 0 :\ \exists y_1, \dots, y_{2^{k-1}} \in Y :\ T(B_X) \subset \cup_{i=1}^{2^{k-1}} (y_i + \varepsilon B_Y)\right\}, \quad k \in \mathbb{N}.$$

Kolmogorov, Tikhomirov, Birman and Solomyak [3, 16, 34] studied properties of ε-entropy (this magnitude is related to entropy numbers of embedding operators).

Denote by l_p^m $(1 \leqslant p \leqslant \infty)$ the space $\mathbb{R}^m$ with norm

$$\|(x_1, \dots, x_m)\|_{l_p^m} = \begin{cases} (|x_1|^p + \dots + |x_m|^p)^{1/p} \text{ for } p < \infty, \\ \max\{|x_1|, \dots, |x_m|\} \text{ for } p = \infty. \end{cases}$$

Estimates for entropy numbers of the embedding operator of l_p^m into l_q^m were obtained in the paper of Schütt [33] (see also [9]). Later Edmunds and Netrusov [7], [8] generalized this result for vector-valued sequence spaces (in particular, for sequence spaces with mixed norm).

Haroske, Triebel, Kühn, Leopold, Sickel and Skrzypczak [10–15, 17–24] studied the problem of estimating entropy numbers of embeddings of weighted sequence spaces or weighted Besov and Triebel–Lizorkin spaces.

Triebel [35] and Mieth [31] studied the problem of estimating entropy numbers of embedding operators of weighted Sobolev spaces on a ball with weights that have singularity at the origin.

Lifshits and Linde [28] obtained estimates for entropy numbers of two-weighted Hardy-type operators on a semiaxis (under some conditions on weights). The similar

problem for one-weighted Riemann-Liouville operators was considered in the paper of Lomakina and Stepanov [29]. In addition, Lifshits and Linde [25–27] studied the problem of estimating entropy numbers of two-weighted summation operators on a tree.

This paper is organized as follows. In §2 we obtain the general result about upper estimates for entropy numbers of embedding operators of function spaces on a set with tree-like structure. The properties of such spaces are almost the same as properties of function spaces defined in [36, 37] (see Assumptions 1, 2, 3), but there are some differences. In particular, here we suppose that (6) holds; this condition cannot be directly derived from known results for weighted Sobolev and Lebesgue spaces. Therefore, first we consider some particular cases of function spaces satisfying Assumptions A–C or A, B, D (see §3) and we prove that these spaces satisfy Assumptions 1–3. In §4 we obtain order estimates for entropy numbers of embedding operators of weighted Sobolev spaces; to this end, we prove that under given conditions on weights Assumptions A–C or A, B, D hold. In §5 we obtain order estimates for entropy numbers of two-weighted summation operators on a tree in critical cases.

2 Upper estimates for entropy numbers of embedding operators of function spaces on sets with tree-like structure

Let us give some notations.

Let $(\Omega, \Sigma, \mathrm{mes})$ be a measure space. We say that sets $A, B \subset \Omega$ are disjoint if $\mathrm{mes}(A\cap B) = 0$. Let $E, E_1, \dots, E_m \subset \Omega$ be measurable sets, and let $m \in \mathbb{N}\cup\{\infty\}$. We say that $\{E_i\}_{i=1}^m$ is a partition of E if the sets E_i are pairwise disjoint and $\mathrm{mes}\left((\cup_{i=1}^m E_i) \triangle E\right) = 0$.

Denote by $\chi_E(\cdot)$ the indicator function of a set E.

Let $\mathcal{G}$ be a graph containing at most countable number of vertices. We shall denote by $\mathbf{V}(\mathcal{G})$ the vertex set of $\mathcal{G}$. Two vertices are called *adjacent* if there is an edge between them. Let $\xi_i \in \mathbf{V}(\mathcal{G})$, $1 \leqslant i \leqslant n$. The sequence $(\xi_1, \dots, \xi_n)$ is called a *path* if the vertices ξ_i and ξ_{i+1} are adjacent for any $i = 1, \dots, n-1$. If all the vertices ξ_i are distinct, then such a path is called *simple*.

Let $(\mathcal{T}, \xi_0)$ be a tree with a distinguished vertex (or a root) ξ_0. We introduce a partial order on $\mathbf{V}(\mathcal{T})$ as follows: we say that $\xi' > \xi$ if there exists a simple path $(\xi_0, \xi_1, \dots, \xi_n, \xi')$ such that $\xi = \xi_k$ for some $k \in \overline{0, n}$. In this case, we set $\rho_{\mathcal{T}}(\xi, \xi') = \rho_{\mathcal{T}}(\xi', \xi) = n + 1 - k$. In addition, we denote $\rho_{\mathcal{T}}(\xi, \xi) = 0$. If $\xi' > \xi$ or $\xi' = \xi$, then we write $\xi' \geqslant \xi$. This partial order on $\mathcal{T}$ induces a partial order on its subtree.

Let $\mathcal{G}$ be a disjoint union of trees $(\mathcal{T}_j, \xi_j)$, $1 \leqslant j \leqslant k$. Then the partial order on each tree $\mathcal{T}_j$ induces the partial order on $\mathcal{G}$.

Given $j \in \mathbb{Z}_+$, $\xi \in \mathbf{V}(\mathcal{T})$, we denote

$$\mathbf{V}_j(\xi) := \mathbf{V}_j^{\mathcal{T}}(\xi) := \{\xi' \geqslant \xi : \ \rho_{\mathcal{T}}(\xi, \xi') = j\}.$$

For $\xi \in \mathbf{V}(\mathcal{T})$ we denote by $\mathcal{T}_\xi = (\mathcal{T}_\xi, \xi)$ the subtree in $\mathcal{T}$ with vertex set

$$\{\xi' \in \mathbf{V}(\mathcal{T}) : \xi' \geqslant \xi\}.$$

Let $\mathcal{G}$ be a subgraph in $\mathcal{T}$. Denote by $\mathbf{V}_{\max}(\mathcal{G})$ and $\mathbf{V}_{\min}(\mathcal{G})$ the sets of maximal and minimal vertices in $\mathcal{G}$, respectively.

Let $\mathbf{W} \subset \mathbf{V}(\mathcal{T})$. We say that $\mathcal{G} \subset \mathcal{T}$ is a maximal subgraph on the vertex set $\mathbf{W}$ if $\mathbf{V}(\mathcal{G}) = \mathbf{W}$ and any two vertices ξ', $\xi'' \in \mathbf{W}$ adjacent in $\mathcal{T}$ are also adjacent in $\mathcal{G}$. Given subgraphs Γ_1, $\Gamma_2 \subset \mathcal{T}$, we denote by $\Gamma_1 \cap \Gamma_2$ the maximal subgraph in $\mathcal{T}$ on the vertex set $\mathbf{V}(\Gamma_1) \cap \mathbf{V}(\Gamma_2)$.

Let $\mathbf{P} = \{\mathcal{T}_j\}_{j\in\mathbb{N}}$ be a family of subtrees in $\mathcal{T}$ such that $\mathbf{V}(\mathcal{T}_j) \cap \mathbf{V}(\mathcal{T}_{j'}) = \varnothing$ for $j \neq j'$ and $\cup_{j\in\mathbb{N}}\mathbf{V}(\mathcal{T}_j) = \mathbf{V}(\mathcal{T})$. Then $\{\mathcal{T}_j\}_{j\in\mathbb{N}}$ is called a partition of the tree $\mathcal{T}$. Let ξ_j be the minimal vertex of $\mathcal{T}_j$. We say that the tree $\mathcal{T}_s$ succeeds the tree $\mathcal{T}_j$ (or $\mathcal{T}_j$ precedes the tree $\mathcal{T}_s$) if $\xi_j < \xi_s$ and

$$\{\xi \in \mathcal{T} : \ \xi_j \leqslant \xi < \xi_s\} \subset \mathbf{V}(\mathcal{T}_j).$$

If $\Gamma \subset \mathcal{T}$ is the maximal subgraph on the vertex set $\mathbf{W}$, we set $\mathbf{P}|_\Gamma = \{\Gamma \cap \mathcal{T}_j\}_{j\in\mathbb{N}}$.

We consider the function spaces on sets with tree-like structure from [36, 37].

Let $(\Omega, \Sigma, \mathrm{mes})$ be a measure space, let $\hat{\Theta}$ be a countable partition of Ω into measurable subsets, let $(\mathcal{A}, \xi_0)$ be a tree such that

$$\exists c_1 \geqslant 1 : \quad \mathrm{card}\, \mathbf{V}_1^{\mathcal{A}}(\xi) \leqslant c_1, \quad \xi \in \mathbf{V}(\mathcal{A}), \tag{1}$$

and let $\hat{F} : \mathbf{V}(\mathcal{A}) \to \hat{\Theta}$ be a bijective mapping.

Throughout we consider at most countable partitions into measurable subsets.

Let $1 \leqslant p$, $q < \infty$ be arbitrary numbers. We suppose that, for any measurable subset $E \subset \Omega$, the following spaces are defined:

- the space $X_p(E)$ with seminorm $\|\cdot\|_{X_p(E)}$,
- the Banach space $Y_q(E)$ with norm $\|\cdot\|_{Y_q(E)}$,

which all satisfy the following conditions:

1. $X_p(\Omega) \subset Y_q(\Omega)$;
2. $X_p(E) = \{f|_E : \ f \in X_p(\Omega)\}$, $Y_q(E) = \{f|_E : \ f \in Y_q(\Omega)\}$;
3. if $\mathrm{mes}\, E = 0$, then $\dim\, Y_q(E) = \dim\, X_p(E) = 0$;

4. if $E \subset \Omega$, $E_j \subset \Omega$ ($j \in \mathbb{N}$) are measurable subsets, $E = \sqcup_{j\in\mathbb{N}} E_j$, then

$$\|f\|_{X_p(E)} = \left\| \left\{ \|f|_{E_j}\|_{X_p(E_j)} \right\}_{j\in\mathbb{N}} \right\|_{l_p}, \quad f \in X_p(E), \tag{2}$$

$$\|f\|_{Y_q(E)} = \left\| \left\{ \|f|_{E_j}\|_{Y_q(E_j)} \right\}_{j\in\mathbb{N}} \right\|_{l_q}, \quad f \in Y_q(E); \tag{3}$$

5. if $E \in \Sigma$, $f \in Y_q(\Omega)$, then $f \cdot \chi_E \in Y_q(\Omega)$.

Let $\mathcal{P}(\Omega) \subset X_p(\Omega)$ be a subspace of finite dimension r_0 and let $\|f\|_{X_p(\Omega)} = 0$ for any $f \in \mathcal{P}(\Omega)$. For each measurable subset $E \subset \Omega$ we write $\mathcal{P}(E) = \{P|_E : \; P \in \mathcal{P}(\Omega)\}$. Let $G \subset \Omega$ be a measurable subset and let T be a partition of G. We set

$$\mathcal{S}_T(\Omega) = \{f : \Omega \to \mathbb{R} : \; f|_E \in \mathcal{P}(E), \; f|_{\Omega\setminus G} = 0\}. \tag{4}$$

If T is finite, then $\mathcal{S}_T(\Omega) \subset Y_q(\Omega)$ (see property 5).

For any finite partition $T = \{E_j\}_{j=1}^n$ of the set E and for each function $f \in Y_q(\Omega)$ we put

$$\|f\|_{p,q,T} = \left(\sum_{j=1}^n \|f|_{E_j}\|_{Y_q(E_j)}^{\sigma_{p,q}} \right)^{\frac{1}{\sigma_{p,q}}} \tag{5}$$

with $\sigma_{p,q} = \min\{p, \, q\}$. Denote by $Y_{p,q,T}(E)$ the space $Y_q(E)$ with the norm $\|\cdot\|_{p,q,T}$. Notice that $\|\cdot\|_{Y_q(E)} \leqslant \|\cdot\|_{p,q,T}$.

For each subtree $\mathcal{A}' \subset \mathcal{A}$ we set

$$\Omega_{\mathcal{A}'} = \cup_{\xi\in\mathbf{V}(\mathcal{A}')} \hat{F}(\xi).$$

In [37] upper estimates for entropy numbers of the embedding operator of the space $\hat{X}_p(\Omega)$ into $Y_q(\Omega)$ were obtained under some conditions on these spaces (the space $\hat{X}_p(\Omega) \cong X_p(\Omega)/\mathcal{P}(\Omega)$ will be defined later). Some limiting relations between the parameters were not considered. Here we investigate one of those critical cases.

Throughout we assume that $1 < p < q < \infty$ and the following conditions hold.

Assumption 1. *There exist a partition $\{\mathcal{A}_{t,i}\}_{t\geqslant t_0, \, i\in\hat{J}_t}$ of the tree $\mathcal{A}$ and a number $c \geqslant 1$ with the following properties.*

1. *If the tree $\mathcal{A}_{t',i'}$ follows the tree $\mathcal{A}_{t,i}$, then $t' = t + 1$.*

2. *For each vertex $\xi_* \in \mathbf{V}(\mathcal{A})$ there exists a linear continuous projection $P_{\xi_*} : Y_q(\Omega) \to \mathcal{P}(\Omega)$ such that for any function $f \in X_p(\Omega)$ and for any subtree $\mathcal{D} \subset \mathcal{A}$ rooted at ξ_**

$$\|f - P_{\xi_*} f\|_{Y_q(\Omega_{\mathcal{D}})}^q \leqslant c \sum_{t=t_0}^\infty \sum_{i\in J_{t,\mathcal{D}}} 2^{\left(1-\frac{q}{p}\right)t} \|f\|_{X_p(\Omega_{\mathcal{D}_{t,i}})}^q; \tag{6}$$

here

$$J_{t,\mathcal{D}} = \{i \in \hat{J}_t : \; \mathbf{V}(\mathcal{A}_{t,i}) \cap \mathbf{V}(\mathcal{D}) \neq \varnothing\}, \quad \mathcal{D}_{t,i} = \mathcal{A}_{t,i} \cap \mathcal{D}. \tag{7}$$

Assumption 2. *There exist numbers $\delta_* > 0$ and $c_2 \geqslant 1$ such that for each vertex $\xi \in \mathbf{V}(\mathcal{A})$ and for any $n \in \mathbb{N}$, $m \in \mathbb{Z}_+$ there exists a partition $T_{m,n}(G)$ of the set $G = \hat{F}(\xi)$ with the following properties:*

1. $\operatorname{card} T_{m,n}(G) \leqslant c_2 \cdot 2^m n$.

2. *For any $E \in T_{m,n}(G)$ there exists a linear continuous operator $P_E : Y_q(\Omega) \to \mathcal{P}(E)$ such that for any function $f \in X_p(\Omega)$*

$$\|f - P_E f\|_{Y_q(E)} \leqslant (2^m n)^{-\delta_*} 2^{\left(\frac{1}{q} - \frac{1}{p}\right)t} \|f\|_{X_p(E)}, \tag{8}$$

where $t \geqslant t_0$ is such that $\xi \in \cup_{j \in \hat{J}_t} \mathbf{V}(\mathcal{A}_{t,j})$.

3. *For any $E \in T_{m,n}(G)$*

$$\operatorname{card} \{E' \in T_{m\pm 1,n}(G) :\, \operatorname{mes}(E \cap E') > 0\} \leqslant c_2. \tag{9}$$

Assumption 3. *There exist numbers $\gamma_* > 0$, $c_3 \geqslant 1$ and an absolutely continuous function $\psi_* : (0, \infty) \to (0, \infty)$ such that $\lim\limits_{y \to \infty} \frac{y\psi_*'(y)}{\psi_*(y)} = 0$ and for $\nu_t := \sum\limits_{i \in \hat{J}_t} \operatorname{card} \mathbf{V}(\mathcal{A}_{t,i})$ the following estimate holds:*

$$\nu_t \leqslant c_3 \cdot 2^{\gamma_* 2^t} \psi_*(2^{2^t}) =: c_3 \overline{\nu}_t, \quad t \geqslant t_0. \tag{10}$$

Assumption 1 together with the inequality $p < q$ implies that for any $t \geqslant t_0$ and for each vertex $\xi_* \in \mathbf{V}^{\mathcal{A}}_{t-t_0}(\xi_0)$ there exists a linear continuous projection $P_{\xi_*} : Y_q(\Omega) \to \mathcal{P}(\Omega)$ such that for any function $f \in X_p(\Omega)$ and for any subtree $\mathcal{D} \subset \mathcal{A}$ rooted at ξ_*

$$\|f - P_{\xi_*} f\|_{Y_q(\Omega_{\mathcal{D}})} \leqslant c \cdot 2^{\left(\frac{1}{q} - \frac{1}{p}\right)t} \|f\|_{X_p(\Omega_{\mathcal{D}})}. \tag{11}$$

Hence, Assumptions 1–3 from [36, 37] hold with $\lambda_* = \mu_* = \frac{1}{p} - \frac{1}{q}$ and $u_* \equiv 1$. In particular, there exist a linear continuous projection $\hat{P} : Y_q(\Omega) \to \mathcal{P}(\Omega)$ and a number $M > 0$ such that for any function $f \in X_p(\Omega)$ the following estimate holds:

$$\|f - \hat{P} f\|_{Y_q(\Omega)} \leqslant M \|f\|_{X_p(\Omega)}.$$

As $\hat{P}$ we take the operator P_{ξ_0} (recall that ξ_0 is the root of $\mathcal{A}$). Similarly as in [37] we set

$$\hat{X}_p(\Omega) = \{f - \hat{P} f :\, f \in X_p(\Omega)\}$$

and denote by I the embedding operator of $\hat{X}_p(\Omega)$ into $Y_q(\Omega)$.

We set $\mathfrak{Z}_0 = (p,\, q,\, c_1,\, c_2,\, c_3,\, c,\, \delta_*,\, \gamma_*,\, \psi_*)$.

We use the following notations for order inequalities. Let X, Y be sets, and let $f_1,\, f_2 :\; X \times Y \to \mathbb{R}_+$. We write $f_1(x,\, y) \underset{y}{\lesssim} f_2(x,\, y)$ (or $f_2(x,\, y) \underset{y}{\gtrsim} f_1(x,\, y)$) if for any $y \in Y$ there exists $c(y) > 0$ such that $f_1(x,\, y) \leqslant c(y) f_2(x,\, y)$ for any $x \in X$; $f_1(x,\, y) \underset{y}{\asymp} f_2(x,\, y)$ if $f_1(x,\, y) \underset{y}{\lesssim} f_2(x,\, y)$ and $f_2(x,\, y) \underset{y}{\lesssim} f_1(x,\, y)$.

Theorem 1. *Suppose that Assumptions 1–3 hold. Then*

$$e_n(I : \hat{X}_p(\Omega) \to Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}. \tag{12}$$

Similarly as in [36], [37] we introduce some more notation.

- $\hat{\xi}_{t,i}$ is the minimal vertex of the tree $\mathcal{A}_{t,i}$.
- $G_t = \cup_{\xi \in \mathbf{V}(\Gamma_t)} \hat{F}(\xi) = \cup_{i \in \hat{J}_t} \Omega_{\mathcal{A}_{t,i}}$.
- $\tilde{\Gamma}_t$ is the maximal subgraph on the vertex set $\cup_{j \geqslant t} \mathbf{V}(\Gamma_j)$, $t \in \mathbb{N}$.
- $\{\tilde{\mathcal{A}}_{t,i}\}_{i \in \overline{J}_t}$ is the set of connected components of the graph $\tilde{\Gamma}_t$.
- $\tilde{U}_{t,i} = \cup_{\xi \in \mathbf{V}(\tilde{\mathcal{A}}_{t,i})} \hat{F}(\xi)$.
- $\tilde{U}_t = \cup_{i \in \overline{J}_t} \tilde{U}_{t,i} = \cup_{\xi \in \mathbf{V}(\tilde{\Gamma}_t)} \hat{F}(\xi)$.

If $t \geqslant t_0$, then

$$\mathbf{V}_{\min}(\tilde{\Gamma}_t) = \mathbf{V}_{\min}(\Gamma_t) = \{\hat{\xi}_{t,i}\}_{i \in \hat{J}_t} \tag{13}$$

(see [36]); hence, we may assume that

$$\overline{J}_t = \hat{J}_t, \quad t \geqslant t_0. \tag{14}$$

The set $\hat{J}_{t_0}$ is a singleton. Denote $\{i_0\} = \hat{J}_{t_0}$.

In [37] the operators Q_t and $P_{t,m}$ were defined as follows.

Definition of the operator Q_t. For each $t \geqslant t_0$, $i \in \hat{J}_t \overset{(14)}{=} \overline{J}_t$ there exists a linear continuous operator $\tilde{P}_{t,i} : Y_q(\Omega) \to \mathcal{P}(\Omega)$ such that for any function $f \in \hat{X}_p(\Omega)$ and for any subtree $\mathcal{A}' \subset \mathcal{A}$ rooted at $\hat{\xi}_{t,i}$

$$\|f - \tilde{P}_{t,i} f\|_{Y_q(\Omega_{\mathcal{A}'})} \overset{(11)}{\underset{\mathfrak{Z}_0}{\lesssim}} 2^{-\left(\frac{1}{p}-\frac{1}{q}\right)t} \|f\|_{X_p(\Omega_{\mathcal{A}'})}. \tag{15}$$

As $\tilde{P}_{t,i}$ we take $P_{\hat{\xi}_{t,i}}$. From the definition of the space $\hat{X}_p(\Omega)$ and of the operator $\hat{P}$ it follows that $\tilde{P}_{t_0,i_0}|_{\hat{X}_p(\Omega)} = 0$ (see [37]).

We set

$$\begin{gathered} Q_t f(x) = \tilde{P}_{t,i} f(x) = P_{\hat{\xi}_{t,i}} f(x) \text{ for } x \in \tilde{U}_{t,i},\ i \in \overline{J}_t, \\ Q_t f(x) = 0 \text{ for } x \in \Omega \backslash \tilde{U}_t, \end{gathered} \tag{16}$$

$$T_t = \{U_{t+1,i}\}_{i \in \overline{J}_{t+1}}. \tag{17}$$

Since $p < q$, we have for any $f \in B\hat{X}_p(\Omega)$

$$\|f - Q_t f\|_{Y_q(\tilde{U}_t)} \leqslant \|f - Q_t f\|_{Y_{p,q,T_{t-1}}(\tilde{U}_t)} \underset{\mathfrak{Z}_0}{\overset{(15)}{\lesssim}} 2^{-\left(\frac{1}{p}-\frac{1}{q}\right)t}. \tag{18}$$

Notice that if $t < t_0$, then $Q_t f = Q_{t+1} f = 0$ (since $\tilde{P}_{t,i_0} = 0$ for $t < t_0$).

Throughout we set $\log x := \log_2 x$.

Definition of the operators $P_{t,m}$**.** For $t \geqslant t_0$ we set

$$m_t = \lceil \log \nu_t \rceil. \tag{19}$$

In [36] for each $m \in \mathbb{Z}_+$ the set $G_{m,t} \subset G_t$, the partition $\tilde{T}_{t,m}$ of the set $G_{m,t}$ and the linear continuous operator

$$P_{t,m} : Y_q(\Omega) \to \mathcal{S}_{\tilde{T}_{t,m}}(\Omega) \tag{20}$$

we constructed. Here the following properties hold:

1. $G_{m,t} \subset G_{m+1,t}$, $G_{m_t,t} = G_t$;
2. for any $m \in \mathbb{Z}_+$
$$\operatorname{card} \tilde{T}_{t,m} \underset{\mathfrak{Z}_0}{\lesssim} 2^m; \tag{21}$$
3. for any function $f \in \hat{X}_p(\Omega)$ and for any set $E \in \tilde{T}_{t,m}$
$$\|f - P_{t,m} f\|_{Y_q(E)} \underset{\mathfrak{Z}_0}{\lesssim} 2^{-\left(\frac{1}{p}-\frac{1}{q}\right)t} \|f\|_{X_p(E)}, \quad m \leqslant m_t, \tag{22}$$
$$\|f - P_{t,m} f\|_{Y_q(E)} \underset{\mathfrak{Z}_0}{\lesssim} 2^{-\left(\frac{1}{p}-\frac{1}{q}\right)t} \cdot 2^{-\delta_*(m-m_t)} \|f\|_{X_p(E)}, \quad m > m_t; \tag{23}$$
4. for any set $E \in \tilde{T}_{t,m}$
$$\operatorname{card} \{E' \in \tilde{T}_{t,m\pm 1} : \ \operatorname{mes}(E \cap E') > 0\} \underset{\mathfrak{Z}_0}{\lesssim} 1. \tag{24}$$

Moreover, we may assume that

$$\tilde{T}_{m_t,t} = \{\hat{F}(\xi)\}_{\xi \in \mathbf{V}(\Gamma_t)}, \tag{25}$$

$$P_{t,m_t} f|_{\hat{F}(\xi)} = P_\xi f|_{\hat{F}(\xi)}, \quad \xi \in \mathbf{V}(\Gamma_t) \tag{26}$$

(it follows from the construction in [36, p. 37–40]).

Let

$$t_*(n) = \min\{t \in \mathbb{N} : \ \overline{\nu}_t \geqslant n\}, \tag{27}$$

$$t_{**}(n) = \min\{t \in \mathbb{N} : \ \overline{\nu}_t \geqslant 2^n\}. \tag{28}$$

Then

$$2^{t_*(n)} \underset{\mathfrak{Z}_0}{\asymp} \log n, \quad 2^{t_{**}(n)} \underset{\mathfrak{Z}_0}{\asymp} n \tag{29}$$

(see [37, formula (49)]).

For any $f \in \hat{X}_p(\Omega)$ the following equality holds:

$$\begin{gathered} f = \sum_{t=t_0}^{t_*(n)-1} (Q_{t+1}f - Q_tf)\chi_{\tilde{U}_{t+1}} + \\ + \sum_{t=t_0}^{t_*(n)-1} \sum_{m=0}^{\infty} (P_{t,m+1}f - P_{t,m}f)\chi_{G_{m,t}} + (f - Q_{t_*(n)}f)\chi_{\tilde{U}_{t_*(n)}} \end{gathered} \tag{30}$$

(it can be proved similarly as formula (82) in [36]).

We set

$$\tilde{Q}f|_{G_t} = P_{t,m_t}f|_{G_t}, \quad t \geqslant t_0, \tag{31}$$

$$\begin{gathered} \tilde{Q}_{n,m}f|_{G_t} = P_{t,m_t+m}f|_{G_t} \quad \text{for} \quad t_*(n) \leqslant t < t_{**}(n), \quad m \in \mathbb{Z}_+, \\ \tilde{Q}_{n,m}f|_{G_t} = 0 \quad \text{for} \quad t < t_*(n) \quad \text{or} \quad t \geqslant t_{**}(n). \end{gathered} \tag{32}$$

Since $\|f - P_{t,m_t}f\|_{Y_q(\hat{F}(\xi))} \overset{(22),(25)}{\underset{\mathfrak{Z}_0}{\lesssim}} 2^{-\left(\frac{1}{p}-\frac{1}{q}\right)t}\|f\|_{X_p(\hat{F}(\xi))}$ for any $\xi \in \mathbf{V}(\Gamma_t)$ and the space $Y_q(\Omega)$ is Banach, we get from the inequality $p < q$ that for any $f \in \hat{X}_p(\Omega)$ the inclusion $\tilde{Q}f \in Y_q(\Omega)$ holds and

$$\|f - \tilde{Q}f\|_{Y_q(\tilde{U}_{t_{**}(n)})} \underset{\mathfrak{Z}_0}{\lesssim} 2^{\left(\frac{1}{q}-\frac{1}{p}\right)t_{**}(n)}\|f\|_{X_p(\Omega)} \overset{(29)}{\underset{\mathfrak{Z}_0}{\lesssim}} n^{\frac{1}{q}-\frac{1}{p}}\|f\|_{X_p(\Omega)}. \tag{33}$$

Further,

$$\begin{gathered} (f - Q_{t_*(n)}f)\chi_{\tilde{U}_{t_*(n)}} = (\tilde{Q}f - Q_{t_*(n)}f)\chi_{\tilde{U}_{t_*(n)}} + \\ + \sum_{m=0}^{\infty} (\tilde{Q}_{n,m+1}f - \tilde{Q}_{n,m}f) + (f - \tilde{Q}f)\chi_{\tilde{U}_{t_{**}(n)}}; \end{gathered} \tag{34}$$

indeed, from (23), (31) and (32) it follows that $\sum\limits_{m=0}^{\infty} (\tilde{Q}_{n,m+1}f - \tilde{Q}_{n,m}f) = (f - \tilde{Q}f)\chi_{\tilde{U}_{t_*(n)} \setminus \tilde{U}_{t_{**}(n)}}$.

Lemma 1. *There exists a sequence* $\{k_t\}_{t=t_0}^{t_*(n)-1} \subset \mathbb{N}$ *such that*

$$\sum_{t=t_0}^{t_*(n)-1} (k_t - 1) \underset{\mathfrak{Z}_0}{\lesssim} n, \tag{35}$$

$$\sum_{t=t_0}^{t_*(n)-1} e_{k_t}(Q_{t+1} - Q_t : \hat{X}_p(\Omega) \to Y_q(\tilde{U}_{t+1})) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}. \tag{36}$$

Lemma 2. *There exists a sequence* $\{k_{t,m}\}_{t_0\leqslant t<t_*(n), m\in\mathbb{Z}_+} \subset \mathbb{N}$ *such that* $\sum\limits_{t,m}(k_{t,m}-1) \underset{\mathfrak{Z}_0}{\lesssim} n$,

$$\sum_{t=t_0}^{t_*(n)-1} \sum_{m=0}^{\infty} e_{k_{t,m}}(P_{t,m+1} - P_{t,m} : \hat{X}_p(\Omega) \to Y_q(G_{m,t})) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}.$$

Lemmas 1 and 2 are proved similarly as Lemmas 6 and 7 in [37].

Lemma 3. *We have*

$$e_n((\tilde{Q} - Q_{t_*(n)})\chi_{\tilde{U}_{t_*(n)}} : \hat{X}_p(\Omega) \to Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}.$$

Notice that for $t_{**}(n) \leqslant t_0$ this estimate follows from (33). Hence, throughout we assume that $t_{**}(n) > t_0$.

We need some auxiliary assertions.

For any $\nu \in \mathbb{N}$ we denote by I_ν the identity operator on $\mathbb{R}^\nu$.

Theorem A. [9,33]. *Let* $1 \leqslant p \leqslant q \leqslant \infty$. *Then*

$$e_k(I_\nu : l_p^\nu \to l_q^\nu) \underset{p,q}{\asymp} \begin{cases} 1, \quad 1 \leqslant k \leqslant \log \nu, \\ \left(\frac{\log\left(1+\frac{\nu}{k}\right)}{k}\right)^{\frac{1}{p}-\frac{1}{q}}, \quad \log \nu \leqslant k \leqslant \nu, \\ 2^{-\frac{k}{\nu}} \nu^{\frac{1}{q}-\frac{1}{p}}, \quad \nu \leqslant k. \end{cases}$$

The following properties of entropy numbers are well-known (see, e.g., [9,32]):

1. if $T : X \to Y$, $S : Y \to Z$ are linear continuous operators, then $e_{k+l-1}(ST) \leqslant e_k(S)e_l(T)$;
2. if T, $S : X \to Y$ are linear continuous operators, then

$$e_{k+l-1}(S + T) \leqslant e_k(S) + e_l(T). \tag{37}$$

In particular, from the first property it follows that

$$e_k(ST) \leqslant \|S\|e_k(T), \quad e_k(ST) \leqslant \|T\|e_k(S). \tag{38}$$

Theorem B. [25]. *Let X, Y be normed spaces, and let $V \in L(X, Y)$, $\{V_\nu\}_{\nu\in\mathcal{N}} \subset L(X, Y)$. Then for any $n \in \mathbb{N}$*

$$e_{n+[\log_2 |\mathcal{N}|]+1}(V) \leqslant \sup_{\nu\in\mathcal{N}} e_n(V_\nu) + \sup_{x\in B_X} \inf_{\nu\in\mathcal{N}} \|Vx - V_\nu x\|_Y.$$

Lemma 4. [39]. *Let $(\mathcal{T}, \xi_*)$ be a tree with finite vertex set, let*

$$\operatorname{card} \mathbf{V}_1(\xi) \leqslant k \quad \textit{for any vertex } \xi \in \mathbf{V}(\mathcal{T}), \tag{39}$$

and let the mapping $\Phi : 2^{\mathbf{V}(\mathcal{T})} \to \mathbb{R}_+$ satisfy the following conditions:

$$\Phi(V_1 \cup V_2) \geqslant \Phi(V_1) + \Phi(V_2), \ V_1, V_2 \subset \mathbf{V}(\mathcal{T}), \ \ V_1 \cap V_2 = \varnothing, \tag{40}$$

$\Phi(\mathbf{V}(\mathcal{T})) > 0$. *Then there is a number $C(k) > 0$ such that for any $n \in \mathbb{N}$ there exists a partition $\mathfrak{S}_n$ of the tree $\mathcal{T}$ into at most $C(k)n$ subtrees $\mathcal{T}_j$, which satisfies the following conditions:*

1. $\Phi(\mathbf{V}(\mathcal{T}_j)) \leqslant \frac{(k+2)\Phi(\mathbf{V}(\mathcal{T}))}{n}$ *for any j such that* $\operatorname{card} \mathbf{V}(\mathcal{T}_j) \geqslant 2$;
2. *if $m \leqslant 2n$, then each element of $\mathfrak{S}_n$ intersects with at most $C(k)$ elements of $\mathfrak{S}_m$.*

Lemma 5. [36]. *Let T be a finite partition of a measurable subset $G \subset \Omega$, $\nu = \dim \mathcal{S}_T(\Omega)$ (see (4)). Then there exists a linear isomorphism $A : \mathcal{S}_T(\Omega) \to \mathbb{R}^\nu$ such that $\|A\|_{Y_{p,q,T}(G)\to l^\nu_{\sigma_{p,q}}} \underset{\sigma_{p,q},r_0}{\lesssim} 1$, $\|A^{-1}\|_{l^\nu_q \to Y_q(G)} \underset{q,r_0}{\lesssim} 1$.*

Lemma 6. [44, formula (60)]. *Let $\Lambda_* : (0, +\infty) \to (0, +\infty)$ be an absolutely continuous function such that $\lim\limits_{y\to+\infty} \frac{y\Lambda_*'(y)}{\Lambda_*(y)} = 0$. Then for any $\varepsilon > 0$*

$$t^{-\varepsilon} \underset{\varepsilon,\Lambda_*}{\lesssim} \frac{\Lambda_*(ty)}{\Lambda_*(y)} \underset{\varepsilon,\Lambda_*}{\lesssim} t^\varepsilon, \quad 1 \leqslant y < \infty, \ \ 1 \leqslant t < \infty. \tag{41}$$

Given $t_*(n) \leqslant t \leqslant t_{**}(n)$, $i \in \hat{J}_t$, we denote

$$\overline{\mathcal{A}}_{t,i} = \begin{cases} \mathcal{A}_{t,i} & \text{for} \quad t < t_{**}(n), \\ \tilde{\mathcal{A}}_{t_{**}(n),i} & \text{for} \quad t = t_{**}(n), \end{cases}$$

$\overline{\Gamma}_t = \Gamma_t$ for $t < t_{**}(n)$, $\overline{\Gamma}_{t_{**}(n)} = \tilde{\Gamma}_{t_{**}(n)}$. Let $\mathcal{D}$ be a subtree in $\tilde{\Gamma}_{t_*(n)}$. We set

$$\hat{J}_{t,\mathcal{D}} = \{i \in \hat{J}_t : \ \mathbf{V}(\overline{\mathcal{A}}_{t,i}) \cap \mathbf{V}(\mathcal{D}) \neq \varnothing\}, \quad \overline{\mathcal{D}}_{t,i} = \overline{\mathcal{A}}_{t,i} \cap \mathcal{D}. \tag{42}$$

Throughout we take as $\varepsilon = \varepsilon(\mathfrak{Z}_0) > 0$ a sufficiently small number (it will be chosen later by $\mathfrak{Z}_0$).

Proof of Lemma 3. We set $t_*'(n) = \max\{t_*(n),\, t_0\}$.

Step 1. Given $t < t_{**}(n)$, we denote by $\hat{\mathcal{A}}_t$ the subtree in $\mathcal{A}$ with vertex set $\mathbf{V}(\mathcal{A})\backslash \mathbf{V}(\tilde{\Gamma}_{t+1})$.

Let $f \in B\hat{X}_p(\Omega)$. For each $t_*'(n) \leqslant t < t_{**}(n)$ we define the mapping $\Phi_{f,t} : 2^{\mathbf{V}(\hat{\mathcal{A}}_t)} \to \mathbb{R}_+$ by

$$\Phi_{f,t}(\mathbf{W}) = \sum_{\xi\in\mathbf{W}\cap\mathbf{V}(\Gamma_t)} \|f\|^p_{X_p(\hat{F}(\xi))}.$$

Then for any disjoint sets $\mathbf{W}_1$, $\mathbf{W}_2$ we have

$$\Phi_{f,t}(\mathbf{W}_1 \sqcup \mathbf{W}_2) = \Phi_{f,t}(\mathbf{W}_1) + \Phi_{f,t}(\mathbf{W}_2). \tag{43}$$

For each $t_*'(n) \leqslant t < t_{**}(n)$ we set

$$\begin{gathered} \varepsilon_t = \sum_{\xi\in\mathbf{V}(\Gamma_t)} \|f\|^p_{X_p(\hat{F}(\xi))}; \\ n_t = \lceil n\cdot 2^{-t}\varepsilon_t\rceil \quad \text{if} \quad \varepsilon_t>0; \quad n_t = 1 \quad \text{if} \quad \varepsilon_t = 0. \end{gathered} \tag{44}$$

Then

$$\sum_{t=t_*'(n)}^{t_{**}(n)-1} \varepsilon_t \leqslant 1, \tag{45}$$

$$\sum_{t=t_*'(n)}^{t_{**}(n)-1} 2^t n_t \leqslant \sum_{t=t_*'(n)}^{t_{**}(n)-1} n\varepsilon_t + \sum_{t=t_*'(n)}^{t_{**}(n)-1} 2^t \overset{(29),(45)}{\leqslant} \hat{C}(\mathfrak{Z}_0)n \tag{46}$$

with $\hat{C}(\mathfrak{Z}_0) \in \mathbb{N}$. By (44),

$$\Phi_{f,t}(\mathbf{V}(\hat{\mathcal{A}}_t)) = \varepsilon_t. \tag{47}$$

Let $f \neq 0$. It follows from Lemma 4 and (1) that there exists a number $C(\mathfrak{Z}_0) \in \mathbb{N}$ and a family of partitions $\{\mathbf{T}_{f,t,l}\}_{0\leqslant l\leqslant \log n_t}$ of the tree $\hat{\mathcal{A}}_t$, which satisfy the following conditions:

$$\operatorname{card} \mathbf{T}_{f,t,l} \leqslant C(\mathfrak{Z}_0)2^{-l}n_t, \tag{48}$$

$$\operatorname{card}\{\mathcal{A}'' \in \mathbf{T}_{f,t,l\pm 1} :\ \mathbf{V}(\mathcal{A}')\cap\mathbf{V}(\mathcal{A}'') \neq \varnothing\} \underset{\mathfrak{Z}_0}{\lesssim} 1, \quad \mathcal{A}' \in \mathbf{T}_{f,t,l}, \tag{49}$$

and for any subtree $\mathcal{A}' \in \mathbf{T}_{f,t,l}$ such that $\operatorname{card}\mathbf{V}(\mathcal{A}') \geqslant 2$

$$\Phi_{f,t}(\mathbf{V}(\mathcal{A}')) \overset{(47)}{\leqslant} C(\mathfrak{Z}_0)2^l n_t^{-1}\varepsilon_t. \tag{50}$$

Moreover, we may assume that

$$\mathbf{T}_{f,t,\lfloor \log n_t \rfloor} = \{\hat{\mathcal{A}}_t\}. \tag{51}$$

For $f \equiv 0$ we set $\mathbf{T}_{f,t,l} = \{\hat{\mathcal{A}}_t\}$.

Given $t < t_{**}(n)$, we denote

$$\mathbf{W}_t = \left\{\xi \in \mathbf{V}(\Gamma_t) : \ \{\xi\} \in \mathbf{T}_{f,t,0}, \ \|f\|^p_{X_p(\hat{F}(\xi))} > C(\mathfrak{Z}_0) n_t^{-1}\varepsilon_t\right\}, \tag{52}$$

$$\overline{\mathbf{W}}_t = \{\xi \in \mathbf{W}_t : \ \mathbf{V}_1^{\mathcal{A}}(\xi) \cap \{\hat{\xi}_{t+1,j}\}_{j\in \hat{J}_{t+1}} \neq \varnothing\}, \tag{53}$$

$$S_t = \{\xi \in \mathbf{V}(\Gamma_t) : \ \exists \mathcal{D} \in \mathbf{T}_{f,t,0} : \ \xi \in \mathbf{V}_{\min}(\mathcal{D})\}, \tag{54}$$

$$\begin{gathered} \hat{S}_t = S_t \cup \left(\cup_{\xi\in\overline{\mathbf{W}}_{t-1}}[\mathbf{V}_1^{\mathcal{A}}(\xi) \cap \mathbf{V}(\Gamma_t)]\right) \quad \text{for} \quad t < t_{**}(n), \\ \hat{S}_{t_{**}(n)} = \cup_{\xi\in\overline{\mathbf{W}}_{t_{**}(n)-1}}[\mathbf{V}_1^{\mathcal{A}}(\xi) \cap \mathbf{V}(\Gamma_{t_{**}(n)})], \end{gathered} \tag{55}$$

$$\hat{S} = \left(\cup_{t=t'_*(n)}^{t_{**}(n)} \hat{S}_t\right) \cup \{\xi_{t'_*(n),j}\}_{j\in \hat{J}_{t'_*(n)}}. \tag{56}$$

Then

$$\hat{S}\backslash \cup_{t=t'_*(n)}^{t_{**}(n)-1} S_t \subset \cup_{t=t'_*(n)}^{t_{**}(n)} \{\hat{\xi}_{t,j} : \ j \in \hat{J}_t\}, \qquad \mathbf{W}_t \subset S_t. \tag{57}$$

For each vertex $\xi \in \hat{S}$ we denote by $\mathcal{D}_{(\xi)}$ the tree with vertex set

$$\mathbf{V}(\mathcal{D}_{(\xi)}) = \{\xi' \geqslant \xi : \ [\xi, \xi'] \cap \hat{S} = \{\xi\}\}. \tag{58}$$

We set

$$\mathbf{T}_f = \{\mathcal{D}_{(\xi)} : \ \xi \in \hat{S}\}. \tag{59}$$

Then $\mathbf{T}_f$ is a partition of the graph $\tilde{\Gamma}_{t'_*(n)}$ into subtrees.

Step 2. We say that $\mathcal{D} \in \tilde{\mathbf{T}}_f$ if $\mathcal{D} \in \mathbf{T}_f$ and for any $\xi \in \mathbf{W}_t$ we have $\mathcal{D} \neq \{\xi\}$.

Let $(\mathcal{D}, \xi_*) \in \tilde{\mathbf{T}}_f$. Then $\xi_* \in \mathbf{V}(\tilde{\Gamma}_{t'_*(n)})$. From Assumption 1 and the inequality $p < q$ it follows that

$$\|f - P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}})} \underset{\mathfrak{Z}_0}{\overset{(6),(42)}{\lesssim}} \sum_{t=t'_*(n)}^{t_{**}(n)} 2^{t\left(1-\frac{q}{p}\right)} \sum_{j\in \hat{J}_{t,\mathcal{D}}} \|f\|^q_{X_p(\Omega_{\overline{\mathcal{D}}_{t,j}})},$$

$$\|f - \tilde{Q}f\|^q_{Y_q(\Omega_{\mathcal{D}})} \overset{(31)}{=} \sum_{t=t'_*(n)}^{t_{**}(n)} \sum_{j\in \hat{J}_{t,\mathcal{D}}} \|f - P_{t,m_t}f\|^q_{Y_q(\Omega_{\overline{\mathcal{D}}_{t,j}})} \underset{\mathfrak{Z}_0}{\overset{(22)}{\lesssim}}$$

$$\lesssim \sum_{t=t'_*(n)}^{t_{**}(n)} 2^{t\left(1-\frac{q}{p}\right)} \sum_{j\in \hat{J}_{t,\mathcal{D}}} \|f\|^q_{X_p(\Omega_{\overline{\mathcal{D}}_{t,j}})}.$$

Hence,

$$\sum_{(\mathcal{D},\,\xi_*)\in \tilde{\mathbf{T}}_f} \|f-P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}})} \underset{\mathfrak{Z}_0}{\lesssim} \sum_{t=t'_*(n)}^{t_{**}(n)} 2^{t\left(1-\frac{q}{p}\right)} \sum_{(\mathcal{D},\,\xi_*)\in \tilde{\mathbf{T}}_f} \sum_{j\in \hat{J}_{t,\mathcal{D}}} \|f\|^q_{X_p(\Omega_{\overline{\mathcal{D}}_{t,j}})}, \qquad (60)$$

$$\sum_{(\mathcal{D},\,\xi_*)\in \tilde{\mathbf{T}}_f} \|f-\tilde{Q}f\|^q_{Y_q(\Omega_{\mathcal{D}})} \underset{\mathfrak{Z}_0}{\lesssim} \sum_{t=t'_*(n)}^{t_{**}(n)} 2^{t\left(1-\frac{q}{p}\right)} \sum_{(\mathcal{D},\,\xi_*)\in \tilde{\mathbf{T}}_f} \sum_{j\in \hat{J}_{t,\mathcal{D}}} \|f\|^q_{X_p(\Omega_{\overline{\mathcal{D}}_{t,j}})}. \qquad (61)$$

Denote by $\tilde{\mathbf{T}}_{f,t}$ the family of trees $\tilde{\mathcal{D}} \in \mathbf{T}_{f,t,0}$ such that

$$\sum_{\xi\in \mathbf{V}(\tilde{\mathcal{D}})\cap \mathbf{V}(\Gamma_t)} \|f\|^p_{X_p(\hat{F}(\xi))} \leqslant C(\mathfrak{Z}_0) n_t^{-1}\varepsilon_t. \qquad (62)$$

Assertion 1. Let $\mathcal{D} \in \tilde{\mathbf{T}}_f$, $t'_*(n) \leqslant t < t_{**}(n)$, $j \in J_{t,\mathcal{D}}$. Then there exists a tree $\mathcal{T} \in \tilde{\mathbf{T}}_{f,t}$ such that $\mathcal{D}_{t,j} = \mathcal{T}_{t,j}$.

Proof of Assertion 1. Let $\mathcal{D} = \mathcal{D}_{(\xi_*)}$, $\xi_* \in \hat{S}$. Since $j \in J_{t,\mathcal{D}}$, the vertices ξ_* and $\hat{\xi}_{t,j}$ are comparable. There exists a tree $\mathcal{T} \in \mathbf{T}_{f,t,0}$ such that

$$\max\{\xi_*, \hat{\xi}_{t,j}\} \in \mathbf{V}(\mathcal{T}). \qquad (63)$$

We claim that $\mathbf{V}(\mathcal{D}_{t,j}) \subset \mathbf{V}(\mathcal{T})$. Indeed, let $\xi \in \mathbf{V}(\mathcal{D}_{t,j})\backslash \mathbf{V}(\mathcal{T})$. Then $\xi > \max\{\hat{\xi}_{t,j},\, \xi_*\}$. We set

$$\eta_* = \min\{\eta \in [\max\{\xi_*,\, \hat{\xi}_{t,j}\},\, \xi] :\; \eta \notin \mathbf{V}(\mathcal{T})\} > \xi_*.$$

Then $\eta_* \in \mathbf{V}(\Gamma_t)$ and there exists a tree $\mathcal{T}' \in \mathbf{T}_{f,t,0}$ such that η_* is the minimal vertex of $\mathcal{T}'$. Hence, $\eta \overset{(54)}{\in} S_t \overset{(55),(56)}{\subset} \hat{S}$; therefore, $\xi \overset{(58)}{\notin} \mathbf{V}(\mathcal{D})$, which leads to a contradiction.

Thus, $\mathbf{V}(\mathcal{T}_{t,j}) \supset \mathbf{V}(\mathcal{D}_{t,j}) \neq \varnothing$.

Let us show that $\mathbf{V}(\mathcal{T}_{t,j}) \subset \mathbf{V}(\mathcal{D}_{t,j})$. Denote by $\hat{\xi}$ the minimal vertex of the tree $\mathcal{T}$. Since $\mathbf{V}(\mathcal{T}_{t,j}) \neq \varnothing$, the vertices $\hat{\xi}$ and $\hat{\xi}_{t,j}$ are comparable. Let us show that $\max\{\xi_*,\, \hat{\xi}_{t,j}\} = \max\{\hat{\xi},\, \hat{\xi}_{t,j}\}$. Indeed, by (63) we have $\max\{\xi_*,\, \hat{\xi}_{t,j}\} \geqslant \max\{\hat{\xi},\, \hat{\xi}_{t,j}\}$. If the inequality is strict, then $\xi_* > \hat{\xi}_{t,j}$. In addition, $\xi_* \in \mathbf{V}(\mathcal{A}_{t,j}) \cap \hat{S}$, and by (54) and (57) we get $\xi_* \in S_t$; i.e., ξ_* is the minimal vertex of some tree from the partition $\mathbf{T}_{f,t,0}$. By (63), $\xi_*{=}\hat{\xi}$.

Thus,

$$\max\{\hat{\xi},\, \hat{\xi}_{t,j}\} = \max\{\xi_*,\, \hat{\xi}_{t,j}\} \in \mathbf{V}(\mathcal{D}_{t,j}). \qquad (64)$$

Suppose that there exists a vertex $\xi \in \mathbf{V}(\mathcal{T}_{t,j})\backslash \mathbf{V}(\mathcal{D})$. We set

$$\eta_{**} = \min\left(\left[\max\{\hat{\xi},\, \hat{\xi}_{t,j}\},\, \xi\right] \backslash \mathbf{V}(\mathcal{D})\right). \tag{65}$$

Then $\eta_{**} > \max\{\hat{\xi},\, \hat{\xi}_{t,j}\}$ and $\eta_{**} \in \mathbf{V}(\mathcal{A}_{t,j})$. Hence, $\eta_{**} \notin \cup_{t'=t'_*(n)}^{t_{**}(n)} \{\hat{\xi}_{t',i} :\ i \in \hat{J}_{t'}\}$. Further, $\eta_{**} \in \mathbf{V}(\mathcal{T}_{t,j})$; therefore, $\eta_{**} \notin \cup_{t'=t'_*(n)}^{t_{**}(n)} S_{t'}$ by (54). From (57) it follows that $\eta_{**} \notin \hat{S}$. This together with (58), (64), (65) yields that $[\xi_*,\, \eta_{**}] \cap \hat{S} = \{\xi_*\}$; applying (58) once again, we get that $\eta_{**} \in \mathbf{V}(\mathcal{D})$, which leads to a contradiction.

It remains to check that $\mathcal{T} \in \tilde{\mathbf{T}}_{f,t}$. If $\mathcal{T} \notin \tilde{\mathbf{T}}_{f,t}$, then $\mathbf{V}(\mathcal{T}) \overset{(50),(62)}{=} \{\xi\}$, $\|f\|_{X_p(\hat{F}(\xi))} > C(\mathfrak{Z}_0) n_t^{-1}\varepsilon_t$; i.e., $\xi \overset{(52)}{\in} \mathbf{W}_t$. Then either $\xi \in \overline{\mathbf{W}}_t$ (in this case, $\mathbf{V}_1^{\mathcal{A}}(\xi) \overset{(54),(55)}{\subset} S_t \cup \hat{S}_{t+1}$) or $\mathbf{V}_1^{\mathcal{A}}(\xi) \overset{(54)}{\subset} S_t$. Hence, $\mathbf{V}_1^{\mathcal{A}}(\xi) \overset{(56)}{\subset} \hat{S}$. This implies that $\mathbf{V}(\mathcal{D}) \overset{(58)}{=} \{\xi\}$ and $\mathcal{D} \notin \tilde{\mathbf{T}}_f$. This completes the proof of Assertion 1. $\diamond$

By (54), the first inclusion of (57), (58) and (59), for any tree $\mathcal{D} \in \mathbf{T}_f$ we have $\overline{\mathcal{D}}_{t_{**}(n),j} = \tilde{\mathcal{A}}_{t_{**}(n),j}$, $j \in J_{t_{**}(n)}$. This together with Assertion 1 yields that

$$\sum_{t=t'_*(n)}^{t_{**}(n)} 2^{t\left(1-\frac{q}{p}\right)} \sum_{(\mathcal{D},\,\xi_*)\in\tilde{\mathbf{T}}_f} \sum_{j\in\hat{J}_{t,\mathcal{D}}} \|f\|^q_{X_p(\Omega_{\overline{\mathcal{D}}_{t,j}})} \leqslant \sum_{t=t'_*(n)}^{t_{**}(n)-1} 2^{t\left(1-\frac{q}{p}\right)} \sum_{\mathcal{T}\in\tilde{\mathbf{T}}_{f,t}} \sum_{j\in J_{t,\mathcal{T}}} \|f\|^q_{X_p(\Omega_{\mathcal{T}_{t,j}})} +$$

$$+2^{t_{**}(n)\left(1-\frac{q}{p}\right)} \sum_{j\in\hat{J}_{t_{**}(n)}} \|f\|^q_{X_p(\Omega_{\tilde{\mathcal{A}}_{t_{**}(n),j}})} \underset{\mathfrak{Z}_0}{\overset{(29)}{\lesssim}} n^{1-\frac{q}{p}} +$$

$$+\sum_{t=t'_*(n)}^{t_{**}(n)-1} 2^{t\left(1-\frac{q}{p}\right)} \sum_{\mathcal{T}\in\tilde{\mathbf{T}}_{f,t}} \left(\sum_{\xi\in\mathbf{V}(\mathcal{T})\cap\mathbf{V}(\Gamma_t)} \|f\|^p_{X_p(\hat{F}(\xi))}\right)^{\frac{q}{p}} \underset{\mathfrak{Z}_0}{\overset{(62)}{\lesssim}}$$

$$\lesssim n^{1-\frac{q}{p}} + \sum_{t=t'_*(n)}^{t_{**}(n)-1} 2^{t\left(1-\frac{q}{p}\right)} \sum_{\mathcal{T}\in\tilde{\mathbf{T}}_{f,t}} \varepsilon_t^{\frac{q}{p}} n_t^{-\frac{q}{p}} \underset{\mathfrak{Z}_0}{\overset{(48)}{\lesssim}} n^{1-\frac{q}{p}} + \sum_{t=t'_*(n)}^{t_{**}(n)-1} 2^{t\left(1-\frac{q}{p}\right)} \varepsilon_t^{\frac{q}{p}} n_t^{1-\frac{q}{p}};$$

i.e.,

$$\sum_{t=t'_*(n)}^{t_{**}(n)} 2^{t\left(1-\frac{q}{p}\right)} \sum_{(\mathcal{D},\,\xi_*)\in\tilde{\mathbf{T}}_f} \sum_{j\in\hat{J}_{t,\mathcal{D}}} \|f\|^q_{X_p(\Omega_{\overline{\mathcal{D}}_{t,j}})} \underset{\mathfrak{Z}_0}{\lesssim} n^{1-\frac{q}{p}} + \sum_{t=t'_*(n)}^{t_{**}(n)-1} 2^{t\left(1-\frac{q}{p}\right)} \varepsilon_t^{\frac{q}{p}} n_t^{1-\frac{q}{p}} =: A. \tag{66}$$

If $n\cdot 2^{-t}\varepsilon_t \geqslant 1$, then $n_t \overset{(44)}{\asymp} n\cdot 2^{-t}\varepsilon_t$; hence,

$$2^{t\left(1-\frac{q}{p}\right)} \varepsilon_t^{\frac{q}{p}} n_t^{1-\frac{q}{p}} \underset{p,q}{\asymp} 2^{t\left(1-\frac{q}{p}\right)} \varepsilon_t^{\frac{q}{p}} n^{1-\frac{q}{p}} 2^{-t\left(1-\frac{q}{p}\right)} \varepsilon_t^{1-\frac{q}{p}} = n^{1-\frac{q}{p}} \varepsilon_t.$$

If $n\cdot 2^{-t}\varepsilon_t<1$, then $n_t=1$. From (45) it follows that $\varepsilon_t\leqslant 1$. Therefore,

$$A \underset{\mathfrak{Z}_0}{\lesssim} n^{1-\frac{q}{p}} + \sum_{t=t'_*(n)}^{t_{**}(n)-1} n^{1-\frac{q}{p}}\varepsilon_t + \sum_{t=t'_*(n)}^{t_{**}(n)-1} 2^{t\left(1-\frac{q}{p}\right)} \overset{(45)}{\underset{\mathfrak{Z}_0}{\lesssim}} n^{1-\frac{q}{p}} + 2^{t_{**}(n)\left(1-\frac{q}{p}\right)} \overset{(29)}{\underset{\mathfrak{Z}_0}{\lesssim}} n^{1-\frac{q}{p}}. \quad (67)$$

From (60), (61), (66), (67) we get that

$$\sum_{(\mathcal{D},\,\xi_*)\in\tilde{\mathbf{T}}_f} (\|f-P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}})} + \|f-\tilde{Q}f\|^q_{Y_q(\Omega_{\mathcal{D}})}) \underset{\mathfrak{Z}_0}{\lesssim} n^{1-\frac{q}{p}};$$

this yields

$$\sum_{(\mathcal{D},\,\xi_*)\in\tilde{\mathbf{T}}_f} \|\tilde{Q}f-P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}})} \underset{\mathfrak{Z}_0}{\lesssim} n^{1-\frac{q}{p}}. \quad (68)$$

Step 3. Let $\mathbf{T}$ be a partition of $\tilde{\Gamma}_{t'_*(n)}$ into subtrees. For each $(\mathcal{D},\,\xi_*)\in\mathbf{T}$ we set

$$P_{\mathbf{T}}f|_{\Omega_{\mathcal{D}}} = P_{\xi_*}f - Q_{t'_*(n)}f. \quad (69)$$

In addition, we put

$$P_{\mathbf{T}}f|_{\Omega\backslash\tilde{U}_{t'_*(n)}} = 0. \quad (70)$$

Then

$$\|\tilde{Q}f - Q_{t'_*(n)}f - P_{\mathbf{T}}f\|^q_{Y_q(\tilde{U}_{t'_*(n)})} = \sum_{(\mathcal{D},\,\xi_*)\in\mathbf{T}} \|\tilde{Q}f-P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}})} \overset{(26),(31)}{=}$$

$$= \sum_{(\mathcal{D},\,\xi_*)\in\mathbf{T},\,\mathrm{card}\,\mathbf{V}(\mathcal{D})\geqslant 2} \|\tilde{Q}f-P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}})}.$$

In particular, from (68) and the definition of $\tilde{\mathbf{T}}_f$ (see the beginning of Step 2) it follows that

$$\|\tilde{Q}f - Q_{t'_*(n)}f - P_{\mathbf{T}_f}f\|^q_{Y_q(\tilde{U}_{t'_*(n)})} \underset{\mathfrak{Z}_0}{\lesssim} n^{1-\frac{q}{p}}. \quad (71)$$

Step 4. Let us define the family $\mathcal{N}$ of partitions of the graph $\tilde{\Gamma}_{t'_*(n)}$ into subtrees. Each of these partitions is constructed as follows. Let $C(\mathfrak{Z}_0)$, $\hat{C}(\mathfrak{Z}_0)$ be as defined at Step 1.

1. We choose the sequence $\{n_t\}_{t=t'_*(n)}^{t_{**}(n)-1}\subset\mathbb{N}$ such that

$$\sum_{t=t'_*(n)}^{t_{**}(n)-1} 2^t n_t \leqslant \hat{C}(\mathfrak{Z}_0)n. \quad (72)$$

2. For each $t \in \{t'_*(n), \dots, t_{**}(n)-1\}$ we take $C(\mathfrak{Z}_0)n_t$ vertices in $\mathbf{V}(\Gamma_t)$ (some of them may coincide); we denote this set by $\mathbf{U}_t$.

3. For each $t \in \{t'_*(n), \dots, t_{**}(n)-1\}$ we choose an arbitrary subset $\hat{\mathbf{V}}_t$ of $\{\xi \in \mathbf{U}_t : \mathbf{V}_1^{\mathcal{A}}(\xi) \cap \mathbf{V}(\Gamma_{t+1}) \neq \varnothing\}$.

4. Let
$$\mathbf{U} = \cup_{j \in \hat{J}_{t'_*(n)}} \{\hat{\xi}_{t'_*(n),j}\} \cup \left(\cup_{t=t'_*(n)}^{t_{**}(n)-1} \mathbf{U}_t\right) \cup \left(\cup_{t=t'_*(n)}^{t_{**}(n)-1} \cup_{\xi \in \hat{\mathbf{V}}_t} [\mathbf{V}_1^{\mathcal{A}}(\xi) \cap \mathbf{V}(\Gamma_{t+1})]\right).$$
This vertex set generates the desired partition $\mathbf{T}$ of the graph $\tilde{\Gamma}_{t'_*(n)}$ into subtrees:
$$\mathbf{T} = \{\mathcal{D}'_{(\xi)} : \xi \in \mathbf{U}\}, \quad \mathbf{V}(\mathcal{D}'_{(\xi)}) = \{\xi' \geqslant \xi : [\xi, \xi'] \cap \mathbf{U} = \{\xi\}\}. \qquad (73)$$

Let us estimate the value $|\mathcal{N}|$.

1. First we estimate the number of choices of $\{n_t\}_{t=t'_*(n)}^{t_{**}(n)-1}$ (we denote this value by N_1). Let $1 \leqslant l \leqslant \hat{C}(\mathfrak{Z}_0)n$. The number of choices of $n_t \in \mathbb{N}$ such that $\sum\limits_{t=t'_*(n)}^{t_{**}(n)-1} 2^t n_t = l$ can be estimated from above by the number of choices of numbers $\hat{n}_t \in \mathbb{N}$ such that $\sum\limits_{t=t'_*(n)}^{t_{**}(n)-1} \hat{n}_t = l$. The last magnitude can be estimated from above by the number of partitions of $\{1, \dots, l\}$ into $t_{**}(n) - t'_*(n)$ intervals. This value does not exceed $(l + t_{**}(n) - t'_*(n))^{t_{**}(n)-t'_*(n)-1}$. Hence,
$$N_1 \leqslant \sum_{1 \leqslant l \leqslant \hat{C}(\mathfrak{Z}_0)n} (l + t_{**}(n) - t'_*(n))^{t_{**}(n)-t'_*(n)-1} \leqslant$$
$$\leqslant \sum_{k=1}^{\hat{C}(\mathfrak{Z}_0)n + t_{**}(n)} k^{t_{**}(n)-1} \underset{\mathfrak{Z}_0}{\lesssim} (\hat{C}(\mathfrak{Z}_0)n + t_{**}(n))^{t_{**}(n)} =: N'_1.$$

2. Given the sequence $\{n_t\}_{t=t'_*(n)}^{t_{**}(n)-1}$, we estimate the number of choices of a set $\cup_{t=t'_*(n)}^{t_{**}(n)-1} \mathbf{U}_t$ (we denote this magnitude by N_2). We have
$$N_2 \overset{(10)}{\leqslant} \prod_{t=t'_*(n)}^{t_{**}(n)-1} (c_3 \overline{\nu}_t)^{C(\mathfrak{Z}_0)n_t} =: N'_2.$$

3. Let us estimate the number of choices of $\hat{\mathbf{V}}_t$, $t'_*(n) \leqslant t \leqslant t_{**}(n)-1$ (denote this value by N_3). Since $\operatorname{card} \mathbf{U}_t \leqslant C(\mathfrak{Z}_0)n_t$, we have
$$N_3 \leqslant \prod_{t=t'_*(n)}^{t_{**}(n)-1} 2^{C(\mathfrak{Z}_0)n_t} =: N'_3.$$

Thus, $|\mathcal{N}| \leqslant N_1'N_2'N_3'$, which yields

$$\log|\mathcal{N}| \leqslant t_{**}(n)\log(\hat{C}(\mathfrak{Z}_0)n + t_{**}(n)) + \sum_{t=t_*'(n)}^{t_{**}(n)-1} C(\mathfrak{Z}_0)n_t\log(c_3\overline{\nu}_t)+$$

$$+\sum_{t=t_*'(n)}^{t_{**}(n)-1} C(\mathfrak{Z}_0)n_t \underset{\mathfrak{Z}_0}{\overset{(10),(29),(41)}{\lesssim}} (\log n)^2 + \sum_{t=t_*'(n)}^{t_{**}(n)-1} 2^t n_t \underset{\mathfrak{Z}_0}{\overset{(72)}{\lesssim}} n;$$

i.e.,

$$\log|\mathcal{N}| \underset{\mathfrak{Z}_0}{\lesssim} n. \tag{74}$$

Step 5. Denote by $\mathcal{N}'$ the family of partitions $\mathbf{T}_f$, $f \in B\hat{X}_p(\Omega)$ (see (59)). Then $\mathcal{N}' \subset \mathcal{N}$ (it follows from (52)–(56), (58), (73) and from the estimates (46), (48)). By (74) and Theorem B, there exists $l_* = l_*(\mathfrak{Z}_0) \in \mathbb{N}$ such that

$$e_{l_*n}(\tilde{Q} - Q_{t_*'(n)} : \hat{X}_p(\Omega) \to Y_q(\tilde{U}_{t_*'(n)})) \leqslant$$

$$\leqslant \sup_{f\in B\hat{X}_p(\Omega)} \inf_{\mathbf{T}\in\mathcal{N}'} \|\tilde{Q}f - Q_{t_*'(n)}f - P_{\mathbf{T}}f\|_{Y_q(\tilde{U}_{t_*'(n)})} + \sup_{\mathbf{T}\in\mathcal{N}'} e_n(P_{\mathbf{T}} : \hat{X}_p(\Omega) \to Y_q(\tilde{U}_{t_*'(n)})) \leqslant$$

$$\leqslant \sup_{f\in B\hat{X}_p(\Omega)} \|\tilde{Q}f - Q_{t_*'(n)}f - P_{\mathbf{T}_f}f\|_{Y_q(\tilde{U}_{t_*'(n)})} + \sup_{\mathbf{T}\in\mathcal{N}'} e_n(P_{\mathbf{T}} : \hat{X}_p(\Omega) \to Y_q(\tilde{U}_{t_*'(n)})) \underset{\mathfrak{Z}_0}{\lesssim}$$

$$\underset{\mathfrak{Z}_0}{\overset{(71)}{\lesssim}} n^{\frac{1}{q}-\frac{1}{p}} + \sup_{\mathbf{T}\in\mathcal{N}'} e_n(P_{\mathbf{T}} : \hat{X}_p(\Omega) \to Y_q(\tilde{U}_{t_*'(n)})).$$

Step 6. It remains to prove that

$$e_n(P_{\mathbf{T}_f} : \hat{X}_p(\Omega) \to Y_q(\tilde{U}_{t_*'(n)})) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}, \tag{75}$$

where $f \in B\hat{X}_p(\Omega)$.

Recall that $\mathbf{T}_f = \{\mathcal{D}_{(\xi)} : \ \xi \in \hat{S}\}$, where $\hat{S}$ is defined by (56) and $\mathcal{D}_{(\xi)}$ is defined by formula (58). Also we observe that if $\xi = \hat{\xi}_{t_*'(n),j}$ for some $j \in \hat{J}_{t_*'(n)}$, then

$$P_{\mathbf{T}_f}|_{\Omega_{\mathcal{D}_{(\xi)}}} \overset{(69)}{=} (P_{\hat{\xi}_{t_*'(n),j}} - Q_{t_*'(n)})|_{\Omega_{\mathcal{D}_{(\hat{\xi}_{t_*'(n),j})}}} \overset{(16)}{=} 0. \tag{76}$$

Let $t_*'(n) \leqslant t < t_{**}(n)$. Recall that the sets S_t and $\overline{\mathbf{W}}_t$ are defined by formulas (54) and (53), respectively. We set

$$\mathbf{V}_{f,t} = S_t \cup \{\xi \in \mathbf{V}(\Gamma_{t+1})\backslash S_{t+1} : \ \exists \eta \in \overline{\mathbf{W}}_t : \ \xi \in \mathbf{V}_1^{\mathcal{A}}(\eta)\}. \tag{77}$$

Notice that $\mathbf{V}_{f,t} \subset \hat{S}$ by (55), (56). Denote by $\hat{\Gamma}_{f,t}$ the maximal subgraph in $\mathcal{A}$ on vertex set

$$\mathbf{V}(\hat{\Gamma}_{f,t}) = \cup_{\xi \in \mathbf{V}_{f,t}} \mathbf{V}(\mathcal{D}_{(\xi)}). \tag{78}$$

We set

$$\Omega_{(f,t)} = \cup_{\xi \in \mathbf{V}(\hat{\Gamma}_{f,t})} \hat{F}(\xi), \quad P_{(f,t)}h = P_{\mathbf{T}_f} h \cdot \chi_{\Omega_{(f,t)}}, \quad h \in Y_q(\Omega). \tag{79}$$

Denote

$$\mathbf{T}'_{f,t} = \{\mathcal{D}_{(\xi)}\}_{\xi \in \mathbf{V}_{f,t}}. \tag{80}$$

Then $\mathbf{T}'_{f,t}$ is a partition of the graph $\hat{\Gamma}_{f,t}$.

We claim that

$$P_{\mathbf{T}_f} = \sum_{t=t'_*(n)}^{t_{**}(n)-1} P_{(f,t)}. \tag{81}$$

Indeed, let $h \in Y_q(\Omega)$. If $x \in \cup_{t=t'_*(n)}^{t_{**}(n)-1} \cup_{\xi \in \mathbf{V}_{f,t}} \Omega_{\mathcal{D}_{(\xi)}}$, then by (79) we get $P_{\mathbf{T}_f} h(x) = \sum\limits_{t=t'_*(n)}^{t_{**}(n)-1} P_{(f,t)}h(x)$. In other cases we have $\sum\limits_{t=t'_*(n)}^{t_{**}(n)-1} P_{(f,t)}h(x) = 0$. On the other hand, if $P_{\mathbf{T}_f} h(x) \ne 0$, then $x \in \Omega_{\mathcal{D}_{(\xi)}}$ for some $\xi \in \hat{S}$ by (59) and (70). From (55), (56), (76) and (77) it follows that $\xi \in \mathbf{V}_{f,t}$ for some $t \in \{t'_*(n), \dots, t_{**}(n)-1\}$. This completes the proof of (81).

Let us prove that there exists a sequence $\{k_t\}_{t'_*(n) \leqslant t < t_{**}(n)}$ such that

$$\sum_{t=t'_*(n)}^{t_{**}(n)-1} (k_t - 1) \underset{\mathfrak{Z}_0}{\lesssim} n,$$

$$\sum_{t=t'_*(n)}^{t_{**}(n)-1} e_{k_t}(P_{(f,t)} : \hat{X}_p(\Omega) \to Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}. \tag{82}$$

Let $\xi \in S_t$. If $\xi \in S_t \backslash \overline{\mathbf{W}}_t$, then we set

$$\tilde{\mathcal{D}}_{(\xi)} = \mathcal{D}_{(\xi)}, \quad \tilde{P}_{(f,t)}|_{\Omega_{\tilde{\mathcal{D}}_{(\xi)}}} = P_{(f,t)}|_{\Omega_{\mathcal{D}_{(\xi)}}} \overset{(69),(79)}{=} (P_\xi - Q_{t'_*(n)})|_{\Omega_{\mathcal{D}_{(\xi)}}}. \tag{83}$$

If $\eta \in \overline{\mathbf{W}}_t$, then we denote by $\tilde{\mathcal{D}}_{(\eta)}$ the tree with vertex set

$$\mathbf{V}(\tilde{\mathcal{D}}_{(\eta)}) = \{\eta\} \cup \left(\cup_{\xi \in \mathbf{V}_1^{\mathcal{A}}(\eta) \cap \mathbf{V}(\Gamma_{t+1}) \backslash S_{t+1}} \mathbf{V}(\mathcal{D}_{(\xi)})\right) \tag{84}$$

and put

$$\tilde{P}_{(f,t)}|_{\Omega_{\tilde{\mathcal{D}}_{(\eta)}}} = (P_\eta - Q_{t'_*(n)})|_{\Omega_{\tilde{\mathcal{D}}_{(\eta)}}}. \tag{85}$$

Observe that by (78) we have

$$\mathbf{T}''_{f,t} := \{\tilde{\mathcal{D}}_{(\xi)}\}_{\xi\in S_t} \quad \text{is the partition of } \hat{\Gamma}_{f,t}. \tag{86}$$

Finally, we set

$$\tilde{P}_{(f,t)}|_{\Omega\setminus\Omega_{(f,t)}} = 0. \tag{87}$$

Let

$$T'_{f,t} = \{\Omega_{\mathcal{D}} :\ \mathcal{D} \in \mathbf{T}'_{f,t}\}, \quad T''_{f,t} = \{\Omega_{\mathcal{D}} :\ \mathcal{D} \in \mathbf{T}''_{f,t}\}.$$

By (69), (79), (83), (85), (87) we have

$$P_{(f,t)} \in \mathcal{S}_{T'_{f,t}}(\Omega), \quad \tilde{P}_{(f,t)} \in \mathcal{S}_{T''_{f,t}}(\Omega). \tag{88}$$

Moreover, the partition $T'_{f,t}$ refines the partition $T''_{f,t}$.

Let $h \in B\hat{X}_p(\Omega)$. Then

$$\|P_{(f,t)}h - \tilde{P}_{(f,t)}h\|^p_{p,q,T'_{f,t}} \overset{(5),(69),(80),(83),(84),(85)}{=}$$

$$= \sum_{\eta\in\overline{\mathbf{W}}_t}\ \sum_{\xi\in\mathbf{V}_1^{\mathcal{A}}(\eta)\cap\mathbf{V}(\Gamma_{t+1})\setminus S_{t+1}} \|P_\xi h - P_\eta h\|^p_{Y_q(\Omega_{\mathcal{D}_{(\xi)}})} \underset{\mathfrak{Z}_0}{\lesssim}$$

$$\lesssim \sum_{\eta\in\overline{\mathbf{W}}_t}\ \sum_{\xi\in\mathbf{V}_1^{\mathcal{A}}(\eta)\cap\mathbf{V}(\Gamma_{t+1})\setminus S_{t+1}} (\|h-P_\xi h\|^p_{Y_q(\Omega_{\mathcal{D}_{(\xi)}})} + \|h - P_\eta h\|^p_{Y_q(\Omega_{\mathcal{D}_{(\xi)}})}) \underset{\mathfrak{Z}_0}{\overset{(11)}{\lesssim}} 2^{-t\left(1-\frac{p}{q}\right)};$$

i.e.,

$$\|P_{(f,t)}h - \tilde{P}_{(f,t)}h\|_{p,q,T'_{f,t}} \underset{\mathfrak{Z}_0}{\lesssim} 2^{-t\left(\frac{1}{p}-\frac{1}{q}\right)}. \tag{89}$$

We set $k'_t = \lceil n\cdot 2^{-\varepsilon(t-t_*(n))}\rceil$, where $\varepsilon > 0$ is a sufficiently small number (it is chosen by $\mathfrak{Z}_0$). For any $t \leqslant t_{**}(n)$ we have

$$\operatorname{card}\mathbf{T}'_{f,t} \overset{(80)}{=} \operatorname{card}\mathbf{V}_{f,t} \underset{\mathfrak{Z}_0}{\overset{(1),(52),(53),(54),(77)}{\lesssim}} \operatorname{card} S_t \underset{\mathfrak{Z}_0}{\overset{(54)}{\lesssim}} \operatorname{card}\mathbf{T}_{f,t,0} \underset{\mathfrak{Z}_0}{\overset{(44),(45),(48)}{\lesssim}} \lceil 2^{-t}n\rceil \underset{\mathfrak{Z}_0}{\overset{(29)}{\asymp}} 2^{-t}n.$$

This together with (38), (88), (89), Theorem A and Lemma 5 implies that for some $c_* = c_*(\mathfrak{Z}_0) > 0$

$$\begin{gathered}\sum_{t=t'_*(n)}^{t_{**}(n)-1} e_{k'_t}(P_{(f,t)} - \tilde{P}_{(f,t)} : \hat{X}_p(\Omega) \to Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\lesssim} \\ \lesssim \sum_{t=t_*(n)}^{t_{**}(n)-1} 2^{-t\left(\frac{1}{p}-\frac{1}{q}\right)}(2^{-t}n)^{\frac{1}{q}-\frac{1}{p}}\cdot 2^{-c_*\cdot 2^{t-\varepsilon(t-t_*(n))}} \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}.\end{gathered} \tag{90}$$

We claim that there exists a sequence $\{k_t''\}_{t_*'(n)\leqslant t<t_{**}(n)}$ such that

$$\sum_{t=t_*'(n)}^{t_{**}(n)-1}(k_t''-1)\underset{\mathfrak{Z}_0}{\lesssim} n,$$

$$\sum_{t=t_*'(n)}^{t_{**}(n)-1}e_{k_t''}(\tilde P_{(f,t)}:\hat X_p(\Omega)\to Y_q(\Omega))\underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}. \tag{91}$$

For each $t_*'(n)\leqslant t<t_{**}(n)$, $0\leqslant l\leqslant \log n_t$ we consider the partition $\mathbf{T}_{f,t,l}$ as defined at Step 1. Let $\mathcal{D}\in\mathbf{T}_{f,t,l}$. We set

$$\mathbf{V}^*_{t,\mathcal{D}}=\{\xi\in\mathbf{V}(\Gamma_t)\cap\mathbf{V}(\mathcal{D}):\ \mathbf{V}_1^{\mathcal{A}}(\xi)\cap\mathbf{V}(\Gamma_{t+1})\neq\varnothing\}$$

and denote by $\mathcal{D}_+$ the subtree in $\mathcal{A}$ with vertex set

$$\mathbf{V}(\mathcal{D}_+)=\mathbf{V}(\mathcal{D})\cup\left(\cup_{\xi\in\mathbf{V}^*_{t,\mathcal{D}}}\cup_{\xi'\in\mathbf{V}_1^{\mathcal{A}}(\xi)\cap\mathbf{V}(\Gamma_{t+1})}\mathbf{V}(\mathcal{A}_{\xi'})\right). \tag{92}$$

Let

$$\tilde{\mathbf{T}}_{f,t,l}=\{\mathcal{D}_+:\ \mathcal{D}\in\mathbf{T}_{f,t,l}\}. \tag{93}$$

By (48),

$$\operatorname{card}\tilde{\mathbf{T}}_{f,t,l}\underset{\mathfrak{Z}_0}{\lesssim} 2^{-l}n_t. \tag{94}$$

We claim that for any tree $\mathcal{D}\in\mathbf{T}_{f,t,l}$

$$\operatorname{card}\{\mathcal{D}'_+:\ \mathcal{D}'\in\mathbf{T}_{f,t,l\pm1}:\ \mathbf{V}(\mathcal{D}_+)\cap\mathbf{V}(\mathcal{D}'_+)\neq\varnothing\}\underset{\mathfrak{Z}_0}{\lesssim}1. \tag{95}$$

Indeed, by (49), it is sufficient to check that if $\mathbf{V}(\mathcal{D}_+)\cap\mathbf{V}(\mathcal{D}'_+)\neq\varnothing$, then $\mathbf{V}(\mathcal{D})\cap\mathbf{V}(\mathcal{D}')\neq\varnothing$. Let $\xi\in\mathbf{V}(\mathcal{D}_+)\cap\mathbf{V}(\mathcal{D}'_+)$. Then either $\xi\in\mathbf{V}(\hat{\mathcal{A}}_t)$ (therefore, $\xi\in\mathbf{V}(\mathcal{D})\cap\mathbf{V}(\mathcal{D}')$) or $\xi\in\mathbf{V}(\mathcal{A}_\eta)$, where $\eta\in\mathbf{V}(\Gamma_t)\cap\mathbf{V}(\mathcal{D})\cap\mathbf{V}(\mathcal{D}')$, $\mathbf{V}_1^{\mathcal{A}}(\eta)\cap\mathbf{V}(\Gamma_{t+1})\neq\varnothing$.

Assertion 2. We have $\tilde{\mathbf{T}}_{f,t,0}|_{\hat\Gamma_{f,t}}=\mathbf{T}''_{f,t}$. If $(\mathcal{T},\hat\xi)\in\mathbf{T}_{f,t,0}$, $\hat\xi\notin\mathbf{V}(\Gamma_t)$, then $\mathbf{V}(\mathcal{T}_+)\cap\mathbf{V}(\hat\Gamma_{f,t})=\varnothing$.

Proof of Assertion 2.

1. Let $(\mathcal{T},\hat\xi)\in\mathbf{T}_{f,t,0}$. We claim that either there exists a tree $\mathcal{D}\in\mathbf{T}''_{f,t}$ such that $\mathbf{V}(\mathcal{T}_+)\cap\mathbf{V}(\hat\Gamma_{f,t})=\mathbf{V}(\mathcal{D})$ or $\mathbf{V}(\mathcal{T}_+)\cap\mathbf{V}(\hat\Gamma_{f,t})=\varnothing$.

   Case $\hat\xi\in\mathbf{V}(\Gamma_t)$. By (54), we have $\hat\xi\in S_t$. Let us show that

$$\mathbf{V}(\mathcal{T}_+)\cap\mathbf{V}(\hat\Gamma_{f,t})=\mathbf{V}(\tilde{\mathcal{D}}_{(\hat\xi)}) \tag{96}$$

   and apply (86).

- We claim that $\mathbf{V}(\mathcal{T}_+) \cap \mathbf{V}(\hat{\Gamma}_{f,t}) \supset \mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})})$. Since $\mathbf{T}''_{f,t}$ is a partition of the graph $\hat{\Gamma}_{f,t}$, we have $\mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})}) \subset \mathbf{V}(\hat{\Gamma}_{f,t})$. Let us prove that $\mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})}) \subset \mathbf{V}(\mathcal{T}_+)$. Indeed, let $\xi \in \mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})})\backslash\mathbf{V}(\mathcal{T}_+)$. We set $\eta_* = \min\{\eta \in [\hat{\xi}, \xi] : \eta \notin \mathbf{V}(\mathcal{T}_+)\} > \hat{\xi}$. Denote by ζ_* the direct predecessor of η_*. Then $\zeta_* \notin \cup_{\zeta\in\mathbf{V}^*_{t,\mathcal{T}}} \cup_{\zeta'\in\mathbf{V}_1^{\mathcal{A}}(\zeta)\cap\mathbf{V}(\Gamma_{t+1})} \mathbf{V}(\mathcal{A}_{\zeta'})$ (otherwise, $\eta \in \mathbf{V}(\mathcal{T}_+)$). Hence, $\zeta_* \in \mathbf{V}(\Gamma_t)$. If $\eta_* \notin \mathbf{V}(\Gamma_t)$, then $\eta_* \in \mathbf{V}(\Gamma_{t+1})$ (by Assumption 1, condition 1); once again, we get $\eta_* \in \mathbf{V}(\mathcal{T}_+)$. Thus, $\eta_* \in \mathbf{V}(\Gamma_t)$ and η_* is the minimal vertex of some tree $\tilde{\mathcal{T}} \in \mathbf{T}_{f,t,0}$. Therefore, $\eta_* \overset{(54)}{\in} S_t \overset{(55),(56)}{\subset} \hat{S}$, which implies $\eta_* \overset{(58),(83),(84)}{\notin} \mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})})$. On the other hand, $\eta_* \in [\hat{\xi}, \xi] \subset \mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})})$, which leads to a contradiction.
- We claim that $\mathbf{V}(\mathcal{T}_+)\cap\mathbf{V}(\hat{\Gamma}_{f,t}) \subset \mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})})$. We have $\mathbf{V}(\hat{\Gamma}_{f,t}) \overset{(86)}{=} \cup_{\eta\in S_t}\mathbf{V}(\tilde{\mathcal{D}}_{(\eta)})$. If $\zeta \in \mathbf{V}(\tilde{\mathcal{D}}_{(\eta)}) \cap \mathbf{V}(\mathcal{T}_+)$, then the vertices $\hat{\xi}$ and η are comparable. Since $\hat{\xi} \in S_t \subset \hat{S}$, the case $\eta < \hat{\xi}$ is impossible by (58), (83), (84). Consequently, $\eta \in [\hat{\xi}, \zeta] \subset \mathbf{V}(\mathcal{T}_+)$; in addition, $\eta \in S_t$. By (54), $\eta \in \mathbf{V}(\Gamma_t)$ is the minimal vertex of some tree in $\mathbf{T}_{f,t,0}$. Therefore, $\eta = \hat{\xi}$; i.e., $\zeta \in \mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})})$.

This completes the proof of (96).

Case $\hat{\xi} \notin \mathbf{V}(\Gamma_t)$. Then $\hat{\xi} \in \mathbf{V}(\Gamma_{t'})$, $t' < t$. Let us check that $\mathbf{V}(\mathcal{T}_+) \cap \mathbf{V}(\hat{\Gamma}_{f,t}) = \varnothing$. Indeed, let $\xi \in \mathbf{V}(\mathcal{T}_+) \cap \mathbf{V}(\hat{\Gamma}_{f,t})$. Then $\xi \overset{(86)}{\in} \mathbf{V}(\tilde{\mathcal{D}}_{(\eta)})$ for some $\eta \in S_t$; i.e., $\eta \overset{(54)}{\in} \mathbf{V}(\Gamma_t)$ is the minimal vertex of some tree from $\mathbf{T}_{f,t,0}$ (this tree does not coincide with $\mathcal{T}$ since $\hat{\xi} \notin \mathbf{V}(\Gamma_t)$ is the minimal vertex of $\mathcal{T}$). In addition, the vertices $\hat{\xi}$ and η are comparable. The case $\eta < \hat{\xi}$ is impossible by Assumption 1 (see condition 1). Hence, $\eta \in [\hat{\xi}, \xi] \subset \mathbf{V}(\mathcal{T}_+)$; i.e., $\eta \in \mathbf{V}(\mathcal{T})$, which leads to a contradiction.

2. Let $\tilde{\mathcal{D}}_{(\hat{\xi})} \in \mathbf{T}''_{f,t}$. We claim that there exists a tree $\mathcal{T} \in \mathbf{T}_{f,t,0}$ such that $\mathbf{V}(\tilde{\mathcal{D}}_{(\hat{\xi})}) = \mathbf{V}(\mathcal{T}_+) \cap \mathbf{V}(\hat{\Gamma}_{f,t})$. Indeed, since $\hat{\xi} \in S_t$ by (86), we have $\hat{\xi} \in \mathbf{V}(\Gamma_t)$ and $\hat{\xi}$ is the minimal vertex of some tree $\mathcal{T} \in \mathbf{T}_{f,t,0}$ (see (54)). By (96), $\mathbf{V}(\mathcal{T}_+) \cap \mathbf{V}(\hat{\Gamma}_{f,t}) = \mathbf{V}(\tilde{\mathcal{D}}_{(\eta)})$ for some $\eta \in S_t$. In addition, $\hat{\xi} \in \mathbf{V}(\mathcal{T}_+) \cap \mathbf{V}(\hat{\Gamma}_{f,t})$; i.e., $\hat{\xi} \in \mathbf{V}(\tilde{\mathcal{D}}_{(\eta)})$ and $\eta = \hat{\xi}$.

This completes the proof of Assertion 2. $\diamond$

Let $\mathcal{D} \in \tilde{\mathbf{T}}_{f,t,0}$, $t'_*(n) \leqslant t < t_{**}(n)$. If the minimal vertex of $\mathcal{D}$ does not belong to $\mathbf{V}(\tilde{\Gamma}_{t'_*(n)})$, by Assertion 2 we have $\mathbf{V}(\mathcal{D})\cap\mathbf{V}(\hat{\Gamma}_{f,t}) = \varnothing$. Hence, if $\mathbf{V}(\mathcal{D})\cap\mathbf{V}(\hat{\Gamma}_{f,t}) \neq \varnothing$, then $\tilde{\Gamma}_{t'_*(n)} \cap \mathcal{D}$ is a tree.

Denote by $\tilde{\mathbf{T}}^n_{f,t,l}$ the partition of $\tilde{\Gamma}_{t'_*(n)}$ formed by connected components of graphs $\tilde{\Gamma}_{t'_*(n)} \cap \mathcal{D}$, $\mathcal{D} \in \tilde{\mathbf{T}}_{f,t,l}$. We have

$$\tilde{P}_{(f,t)}h = P_{\tilde{\mathbf{T}}^n_{f,t,0}}h \cdot \chi_{\Omega_{(f,t)}}, \quad h \in Y_q(\Omega). \tag{97}$$

It follows from (69), (79), (83), (85), (87), (88) and Assertion 2.

Given $0 \leqslant l \leqslant \log n_t$, we set

$$
\begin{gathered}
k_{t,l} = \lceil n \cdot 2^{-\varepsilon(t-t_*(n)+l)} \rceil, \quad t_*(n) \leqslant t < t_{**}(n), \\
k''_t = 1 + \sum_{0\leqslant l \leqslant \log n_t} (k_{t,l}-1), \quad t'_*(n) \leqslant t < t_{**}(n).
\end{gathered} \tag{98}
$$

Then $k''_t - 1 \underset{\mathfrak{Z}_0}{\lesssim} n \cdot 2^{-\varepsilon(t-t_*(n))}$, $\sum\limits_{t=t'_*(n)}^{t_{**}(n)-1} (k''_t - 1) \underset{\mathfrak{Z}_0}{\lesssim} n$.

From (51), (92) and (93) it follows that $\tilde{\mathbf{T}}_{f,t,\lfloor \log n_t \rfloor} = \{\mathcal{A}\}$, $\tilde{\mathbf{T}}^n_{f,t,\lfloor \log n_t \rfloor} = \{\mathcal{A}_{t'_*(n),j}\}_{j\in \hat{J}_{t'_*(n)}}$. From (16) and (69) we get $P_{\tilde{\mathbf{T}}^n_{f,t,\lfloor \log n_t \rfloor}} = 0$. Hence, by (97) and (37),

$$
\begin{gathered}
e_{k''_t}(\tilde{P}_{(f,t)} : \hat{X}_p(\Omega) \to Y_q(\Omega)) \leqslant \\
\leqslant \sum_{0\leqslant l < \lfloor \log n_t \rfloor} e_{k_{t,l}}(P_{\tilde{\mathbf{T}}^n_{f,t,l}} - P_{\tilde{\mathbf{T}}^n_{f,t,l+1}} : \hat{X}_p(\Omega) \to Y_q(\Omega_{(f,t)})).
\end{gathered} \tag{99}
$$

Denote by $\hat{\mathbf{T}}^n_{f,t,l}$ the partition formed by trees $\mathcal{D}' \cap \mathcal{D}''$, where $\mathcal{D}' \in \tilde{\mathbf{T}}^n_{f,t,l}$, $\mathcal{D}'' \in \tilde{\mathbf{T}}^n_{f,t,l+1}$, and either $\mathcal{D}' \in \tilde{\mathbf{T}}_{f,t,l}$ or $\mathcal{D}'' \in \tilde{\mathbf{T}}_{f,t,l+1}$. We set

$$
\tilde{T}^n_{f,t,l} = \{\Omega_{\mathcal{D}}\}_{\mathcal{D}\in\tilde{\mathbf{T}}^n_{f,t,l}}, \quad \hat{T}^n_{f,t,l} = \{E = \Omega_{\mathcal{D}} \cap \Omega_{(f,t)} : \ \mathrm{mes}\, E > 0\}_{\mathcal{D}\in\hat{\mathbf{T}}^n_{f,t,l}}. \tag{100}
$$

Let $h \in B\hat{X}_p(\Omega)$. We show that

$$
(P_{\tilde{\mathbf{T}}^n_{f,t,l}} h - P_{\tilde{\mathbf{T}}^n_{f,t,l+1}} h)\chi_{\Omega_{(f,t)}} \in \mathcal{S}_{\hat{T}^n_{f,t,l}}(\Omega). \tag{101}
$$

Indeed, if $\mathcal{D} = \mathcal{D}' \cap \mathcal{D}''$, $\mathcal{D}' \in \tilde{\mathbf{T}}^n_{f,t,l}$, $\mathcal{D}'' \in \tilde{\mathbf{T}}^n_{f,t,l+1}$, then $P_{\tilde{\mathbf{T}}^n_{f,t,l}} h|_{\Omega_{\mathcal{D}'}} \overset{(69)}{\in} \mathcal{P}(\Omega_{\mathcal{D}'})$, $P_{\tilde{\mathbf{T}}^n_{f,t,l+1}} h|_{\Omega_{\mathcal{D}''}} \overset{(69)}{\in} \mathcal{P}(\Omega_{\mathcal{D}''})$; therefore,

$$
(P_{\tilde{\mathbf{T}}^n_{f,t,l}} h - P_{\tilde{\mathbf{T}}^n_{f,t,l+1}} h)|_{\Omega_{\mathcal{D}}\cap\Omega_{(f,t)}} \in \mathcal{P}(\Omega_{\mathcal{D}} \cap \Omega_{(f,t)}). \tag{102}
$$

We show that if $\mathcal{D}' \notin \tilde{\mathbf{T}}_{f,t,l}$ and $\mathcal{D}'' \notin \tilde{\mathbf{T}}_{f,t,l+1}$, then $(P_{\tilde{\mathbf{T}}^n_{f,t,l}} h - P_{\tilde{\mathbf{T}}^n_{f,t,l+1}} h)|_{\Omega_{\mathcal{D}}\cap\Omega_{(f,t)}} = 0$. Indeed, in this case minimal vertices of the trees $\mathcal{D}'$ and $\mathcal{D}''$ are equal and coincide with $\hat{\xi}_{t_*(n),j}$ for some $j \in \hat{J}_{t_*(n)}$. From (69) it follows that $(P_{\tilde{\mathbf{T}}^n_{f,t,l}} h - P_{\tilde{\mathbf{T}}^n_{f,t,l+1}} h)|_{\Omega_{\mathcal{D}}} = 0$. From (4) and (102) we get (101).

We claim that for any $E' \in \tilde{T}^n_{f,t,l}$, $E'' \in \tilde{T}^n_{f,t,l+1}$

$$
\mathrm{card}\,\{E \in \hat{T}^n_{f,t,l} : \quad E \subset E'\} \underset{\mathfrak{Z}_0}{\lesssim} 1, \quad \mathrm{card}\,\{E \in \hat{T}^n_{f,t,l} : \quad E \subset E''\} \underset{\mathfrak{Z}_0}{\lesssim} 1. \tag{103}
$$

Let us check the first inequality (the second one is proved similarly). Let $E = \Omega_{\mathcal{D}'}$, $\mathcal{D}' \in \tilde{\mathbf{T}}^n_{f,t,l}$, $E \in \hat{T}^n_{f,t,l}$, $E \subset E'$. Since $\tilde{\mathbf{T}}^n_{f,t,l}$ is a partition, we have $E = \Omega_{\mathcal{D}'\cap\mathcal{D}''} \cap \Omega_{(f,t)}$, $\mathcal{D}'' \in \tilde{\mathbf{T}}^n_{f,t,l+1}$, $\mathbf{V}(\mathcal{D}') \cap \mathbf{V}(\mathcal{D}'') \neq \varnothing$. There exist trees $\mathcal{T}' \in \tilde{\mathbf{T}}_{f,t,l}$ and $\mathcal{T}'' \in \tilde{\mathbf{T}}_{f,t,l+1}$

such that $\mathcal{D}'$ and $\mathcal{D}''$ are connected components of the graphs $\mathcal{T}'\cap\tilde{\Gamma}_{t'_*(n)}$ and $\mathcal{T}''\cap\tilde{\Gamma}_{t'_*(n)}$, respectively. Observe that the connected component of the graph $\mathcal{T}''\cap\tilde{\Gamma}_{t'_*(n)}$ whose vertex set intersects with $\mathbf{V}(\mathcal{D}')$ is unique. This together with (95) implies (103).

From (94), (103) and the definition of $\hat{\mathbf{T}}^n_{f,t,l}$ we get that

$$\operatorname{card}\hat{T}^n_{f,t,l}\underset{\mathfrak{Z}_0}{\lesssim}2^{-l}n_t. \tag{104}$$

For any $h\in B\hat{X}_p(\Omega)$ we have

$$\|(P_{\tilde{\mathbf{T}}^n_{f,t,l}}h-P_{\tilde{\mathbf{T}}^n_{f,t,l+1}}h)\chi_{\Omega_{(f,t)}}\|_{p,q,\hat{T}^n_{f,t,l}}\overset{(5),(103)}{\underset{\mathfrak{Z}_0}{\lesssim}}$$

$$\lesssim\|(h-Q_{t'_*(n)}h-P_{\tilde{\mathbf{T}}^n_{f,t,l}}h)\chi_{\tilde{U}_{t'_*(n)}}\|_{p,q,\tilde{T}^n_{f,t,l}}+$$

$$+\|(h-Q_{t'_*(n)}h-P_{\tilde{\mathbf{T}}^n_{f,t,l+1}}h)\chi_{\tilde{U}_{t'_*(n)}}\|_{p,q,\tilde{T}^n_{f,t,l+1}}\overset{(5),(69),(100)}{=}$$

$$\lesssim\left(\sum_{(\mathcal{D},\xi)\in\tilde{\mathbf{T}}^n_{f,t,l}}\|h-P_\xi h\|^p_{Y_q(\Omega_{\mathcal{D}})}\right)^{1/p}+\left(\sum_{(\mathcal{D},\xi)\in\tilde{\mathbf{T}}^n_{f,t,l+1}}\|h-P_\xi h\|^p_{Y_q(\Omega_{\mathcal{D}})}\right)^{1/p}\overset{(11)}{\underset{\mathfrak{Z}_0}{\lesssim}}$$

$$\lesssim 2^{\left(\frac{1}{q}-\frac{1}{p}\right)t'_*(n)}\underset{\mathfrak{Z}_0}{\lesssim}2^{\left(\frac{1}{q}-\frac{1}{p}\right)t_*(n)}\overset{(29)}{\underset{\mathfrak{Z}_0}{\asymp}}(\log n)^{\frac{1}{q}-\frac{1}{p}};$$

i.e.,

$$\|(P_{\tilde{\mathbf{T}}^n_{f,t,l}}h-P_{\tilde{\mathbf{T}}^n_{f,t,l+1}}h)\chi_{\Omega_{(f,t)}}\|_{p,q,\hat{T}^n_{f,t,l}}\underset{\mathfrak{Z}_0}{\lesssim}(\log n)^{\frac{1}{q}-\frac{1}{p}}. \tag{105}$$

From (38), (101), (104), (105) and Lemma 5 we get that

$$\begin{gathered}e_{k_{t,l}}(P_{\tilde{\mathbf{T}}^n_{f,t,l}}-P_{\tilde{\mathbf{T}}^n_{f,t,l+1}}:\hat{X}_p(\Omega)\to Y_q(\Omega_{(f,t)}))\underset{\mathfrak{Z}_0}{\lesssim}\\ \lesssim(\log n)^{\frac{1}{q}-\frac{1}{p}}e_{k_{t,l}}(I_{s_{t,l}}:l_p^{s_{t,l}}\to l_q^{s_{t,l}})=:A_{t,l},\end{gathered} \tag{106}$$

where $s_{t,l}\in\mathbb{N}$,

$$s_{t,l}\leqslant C_*\cdot 2^{-l}n_t\overset{(44),(45)}{\leqslant}C_*\cdot 2^{-l}\lceil n\cdot 2^{-t}\rceil,\quad C_*=C_*(\mathfrak{Z}_0)\geqslant 1.$$

For each $t\leqslant t_{**}(n)$ we have

$$\frac{s_{t,l}}{k_{t,l}}\overset{(98)}{\leqslant}\frac{C_*\cdot 2^{-l}\lceil n\cdot 2^{-t}\rceil}{n\cdot 2^{-\varepsilon(l+t-t_*(n))}}=:\sigma'_{t,l},$$

$$\sigma'_{t,l} \underset{\mathfrak{Z}_0}{\overset{(29)}{\asymp}} \frac{2^{-l}n\cdot 2^{-t}}{n\cdot 2^{-\varepsilon(l+t-t_*(n))}} = 2^{-l(1-\varepsilon)-t(1-\varepsilon)-\varepsilon t_*(n)} \leqslant 1. \tag{107}$$

The sequence $\{\sigma'_{t,l}\}_{l\in\mathbb{Z}_+}$ decreases not slower than some geometric progression. This together with Theorem A implies that there exists $\gamma_0 = \gamma_0(\mathfrak{Z}_0) > 0$ such that

$$\sum_{t=t'_*(n)}^{t_{**}(n)-1} \sum_{0\leqslant l<\log n_t} A_{t,l} \underset{\mathfrak{Z}_0}{\overset{(106)}{\lesssim}} (\log n)^{\frac{1}{q}-\frac{1}{p}} \sum_{t=t'_*(n)}^{t_{**}(n)-1} \sum_{0\leqslant l<\log n_t} s_{t,l}^{\frac{1}{q}-\frac{1}{p}} 2^{-\frac{k_{t,l}}{s_{t,l}}} \underset{\mathfrak{Z}_0}{\lesssim}$$

$$\lesssim (\log n)^{\frac{1}{q}-\frac{1}{p}} \sum_{t=t'_*(n)}^{t_{**}(n)-1} \sum_{0\leqslant l<\log n_t} k_{t,l}^{\frac{1}{q}-\frac{1}{p}} \cdot (\sigma'_{t,l})^{\frac{1}{q}-\frac{1}{p}} \cdot 2^{-\frac{1}{\sigma'_{t,l}}} \underset{\mathfrak{Z}_0}{\overset{(98),(107)}{\lesssim}}$$

$$\lesssim (\log n)^{\frac{1}{q}-\frac{1}{p}} \sum_{t=t_*(n)}^{t_{**}(n)-1} k_{t,0}^{\frac{1}{q}-\frac{1}{p}} \cdot 2^{(t(1-\varepsilon)+\varepsilon t_*(n))\left(\frac{1}{q}-\frac{1}{p}\right)} \cdot 2^{-\gamma_0\cdot 2^{t(1-\varepsilon)+\varepsilon t_*(n)}} \underset{\mathfrak{Z}_0}{\overset{(98)}{\lesssim}}$$

$$\lesssim (\log n)^{\frac{1}{q}-\frac{1}{p}} \cdot n^{\frac{1}{q}-\frac{1}{p}} \cdot 2^{\left(\frac{1}{q}-\frac{1}{p}\right)t_*(n)} \cdot 2^{-\gamma_0\cdot 2^{t_*(n)}} \underset{\mathfrak{Z}_0}{\overset{(29)}{\lesssim}} n^{\frac{1}{q}-\frac{1}{p}};$$

i.e.,

$$\sum_{t=t'_*(n)}^{t_{**}(n)-1} \sum_{0\leqslant l<\log n_t} A_{t,l} \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}. \tag{108}$$

From (99), (106) and (108) we get (91). This together with (90) yields (82). Applying (81), we have (75). Taking into account the estimate obtained at Step 5, we complete the proof of Lemma 3. □

It remains to prove that

$$e_n\left(\sum_{m=0}^{\infty}(\tilde{Q}_{n,m+1}-\tilde{Q}_{n,m}) : \hat{X}_p(\Omega)\to Y_q(\Omega)\right) \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}. \tag{109}$$

In [23] there were obtained order estimates for entropy numbers of diagonal operators with weights of logarithmic type. First we give some notations.

Denote by Φ_0 the class of non-decreasing functions $\varphi : [1,\infty)\to(0,\infty)$ that satisfy the following condition: there exist $c>0$ and $\alpha>0$ such that for any $1\leqslant s\leqslant t<\infty$

$$\frac{\varphi(t)}{\varphi(s)} \leqslant c\left(\frac{1+\log t}{1+\log s}\right)^{\alpha}. \tag{110}$$

We set $w_\varphi(s)=1$ for $0\leqslant s\leqslant 1$ and $w_\varphi(s)=\varphi(s)$ for $s>1$.

Let $\delta > 0$, $w : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, and let $0 < r$, $p \leqslant \infty$. For $x = (x_{m,k})_{m,k\in\mathbb{Z}_+}$ we set

$$\|x|l_r(2^{\delta m}l_p(w))\| := \left(\sum_{m=0}^{\infty} 2^{m\delta r}\left(\sum_{k\in\mathbb{Z}_+}|x_{m,k}w(2^{-m}k)|^p\right)^{\frac{r}{p}}\right)^{\frac{1}{r}}$$

(appropriately modified if $p = \infty$ or $r = \infty$). By $l_r(2^{\delta m}l_p(w))$ we denote the space of sequences x such that $\|x|l_r(2^{\delta m}l_p(w))\| < \infty$.

Theorem C. [23, p. 11]. *Let $0 < p < q \leqslant \infty$, $0 < r$, $s \leqslant \infty$, $\delta > 0$, $\varphi \in \Phi_0$, and let (110) hold with $\alpha = \frac{1}{p} - \frac{1}{q}$. Then*

$$e_n(\mathrm{id} : l_r(2^{\delta m}l_p(w_\varphi)) \to l_s(l_q)) \underset{p,q,r,s,\varphi}{\asymp} \frac{1}{\varphi(2^n)}.$$

Let us prove (109).

Given $m \geqslant m_t$, we denote by $\hat{T}_{t,m}$ the partition of the set G_t formed by $E' \cap E''$, $E' \in \tilde{T}_{t,m}$, $E'' \in \tilde{T}_{t,m+1}$ (the partitions $\tilde{T}_{t,m}$ are defined at page 7). Let $s''_{t,m} = \dim \mathcal{S}_{\hat{T}_{t,m}}(\Omega)$. Then there exists $M = M(\mathfrak{Z}_0) \geqslant 1$ such that

$$s''_{t,m} \overset{(10),(19),(21),(24)}{\leqslant} M \cdot 2^{m-m_t} \cdot 2^{\gamma_* 2^t}\psi_*(2^{2^t}). \tag{111}$$

Given $t \geqslant 0$, $m' \geqslant 0$, we set

$$\hat{s}_{t,m'} = \lceil M \cdot 2^{m'} \cdot 2^{\gamma_* 2^t}\psi_*(2^{2^t})\rceil, \quad s^*_{t,m'} = \sum_{l=0}^{t-1} \hat{s}_{l,m'}.$$

Denote $\varphi(x) = (\log(2 + x))^{\frac{1}{p}-\frac{1}{q}}$, $x \geqslant 0$. Let $k = s^*_{t,m'} + j$, $1 \leqslant j \leqslant \hat{s}_{t,m'}$. Then

$$2^{\gamma_* 2^{t-1}}\psi_*(2^{2^{t-1}}) \underset{\mathfrak{Z}_0}{\lesssim} 2^{-m'}k \underset{\mathfrak{Z}_0}{\lesssim} 2^{\gamma_* 2^t}\psi_*(2^{2^t}), \quad t \geqslant 1,$$

$2^{-m'}k \underset{\mathfrak{Z}_0}{\lesssim} 1$ if $t = 0$. Hence,

$$w_\varphi(2^{-m'}k) \underset{\mathfrak{Z}_0}{\overset{(41)}{\asymp}} 2^{\left(\frac{1}{p}-\frac{1}{q}\right)t}, \quad k = s^*_{t,m'} + j, \quad 1 \leqslant j \leqslant \hat{s}_{t,m'}. \tag{112}$$

From Lemma 5 it follows that there is an isomorphism $\overline{A}_{t,m} : \mathcal{S}_{\hat{T}_{t,m}}(\Omega) \to \mathbb{R}^{s''_{t,m}}$ such that

$$\|\overline{A}_{t,m}\|_{Y_{p,q,\hat{T}_{t,m}}(G_t)\to l_p^{s''_{t,m}}} \underset{\mathfrak{Z}_0}{\lesssim} 1, \quad \|\overline{A}_{t,m}^{-1}\|_{l_q^{s''_{t,m}}\to Y_q(G_t)} \underset{\mathfrak{Z}_0}{\lesssim} 1. \tag{113}$$

Let us define the operator $\overline{A} : \hat{X}_p(\Omega) \to l_\infty(2^{\delta_* m'} l_p(w_\varphi))$ as follows. Consider a function $f \in \hat{X}_p(\Omega)$. Then $(\tilde{Q}_{n,m'+1}f - \tilde{Q}_{n,m'}f)|_{G_t} \overset{(20),(32)}{\in} \mathcal{S}_{\hat{T}_{t,m_t+m'}}(\Omega)$. Let

$$\overline{A}_{t,m_t+m'}((\tilde{Q}_{n,m'+1}f - \tilde{Q}_{n,m'}f)|_{G_t}) = (c_{m',t,j})_{j=1}^{s''_{t,m_t+m'}}.$$

We set

$$(\overline{A}f)_{m',s^*_{t,m'}+j} = \begin{cases} c_{m',t,j} \quad \text{for} \quad 1 \leqslant j \leqslant s''_{t,m_t+m'},\ t \geqslant t_0, \\ 0 \quad \text{for} \quad s''_{t,m_t+m'} + 1 \leqslant j \leqslant \hat{s}_{t,m'} \text{ or } t < t_0. \end{cases}$$

Then

$$\|\overline{A}f\|_{l_\infty(2^{\delta_* m'} l_p(w_\varphi))} = \sup_{m' \geqslant 0} 2^{\delta_* m'} \left(\sum_{k \in \mathbb{Z}_+} |w_\varphi(2^{-m'}k)(\overline{A}f)_{m',k}|^p \right)^{\frac{1}{p}} \underset{\mathfrak{Z}_0}{\overset{(112)}{\lesssim}}$$

$$\lesssim \sup_{m' \geqslant 0} 2^{\delta_* m'} \left(\sum_{t \geqslant t_0} 2^{\left(1-\frac{p}{q}\right)t} \sum_{j=1}^{s''_{t,m_t+m'}} |c_{m',t,j}|^p \right)^{1/p} =: N.$$

From (22), (23) and (24) it follows that for $m \geqslant m_t$

$$\|P_{t,m+1} - P_{t,m}\|_{\hat{X}_p(G_t) \to Y_{p,q,\hat{T}_{t,m}}(G_t)} \underset{\mathfrak{Z}_0}{\lesssim} 2^{-\delta_*(m-m_t)} \cdot 2^{\left(\frac{1}{q}-\frac{1}{p}\right)t} \|f\|_{X_p(G_t)}.$$

Hence, by (32), (113) we get

$$\sum_{j=1}^{s''_{t,m_t+m'}} |c_{m',t,j}|^p \underset{\mathfrak{Z}_0}{\lesssim} 2^{-\delta_* p m'} 2^{\left(\frac{p}{q}-1\right)t} \|f\|^p_{X_p(G_t)};$$

therefore,

$$N \underset{\mathfrak{Z}_0}{\lesssim} \left(\sum_{t \geqslant t_0} \|f\|^p_{X_p(G_t)} \right)^{1/p} = \|f\|_{X_p(\Omega)}.$$

Thus,

$$\|\overline{A}\|_{\hat{X}_p(\Omega) \to l_\infty(2^{\delta_* m'} l_p(w_\varphi))} \underset{\mathfrak{Z}_0}{\lesssim} 1. \tag{114}$$

Let us define the operator $\overline{K} : l_1(l_q) \to Y_q(\Omega)$ by formula

$$\overline{K}((c_{m',s^*_{t,m'}+j})_{m' \in \mathbb{Z}_+, t \in \mathbb{Z}_+, 1 \leqslant j \leqslant \hat{s}_{t,m'}}) = \sum_{m' \in \mathbb{Z}_+} \sum_{t \geqslant t_0} \overline{A}^{-1}_{t,m_t+m'}((c_{m',s^*_{t,m'}+j})_{j=1}^{s''_{t,m_t+m'}}).$$

Since the sets G_t do not overlap pairwise, we have

$$\|\overline{K}((c_{m',s^*_{t,m'}+j})_{m'\in\mathbb{Z}_+,t\in\mathbb{Z}_+,1\leqslant j\leqslant\hat{s}_{t,m'}})\|_{Y_q(\Omega)} \leqslant$$

$$\leqslant \sum_{m'=0}^{\infty}\left(\sum_{t=t_0}^{\infty}\|\overline{A}^{-1}_{t,m_t+m'}((c_{m',s^*_{t,m'}+j})_{j=1}^{s''_{t,m_t+m'}})\|^q_{Y_q(G_t)}\right)^{1/q} \underset{\mathfrak{Z}_0}{\overset{(113)}{\lesssim}}$$

$$\lesssim \sum_{m'=0}^{\infty}\left(\sum_{t\in\mathbb{Z}_+}\sum_{j=1}^{\hat{s}_{t,m'}}|c_{m',s^*_{t,m'}+j}|^q\right)^{1/q} = \|(c_{m',s^*_{t,m'}+j})_{m'\in\mathbb{Z}_+,t\in\mathbb{Z}_+,1\leqslant j\leqslant\hat{s}_{t,m'}}\|_{l_1(l_q)}.$$

Hence,

$$\|\overline{K}\|_{l_1(l_q)\to Y_q(\Omega)} \underset{\mathfrak{Z}_0}{\lesssim} 1. \tag{115}$$

Let $\mathrm{id} : l_\infty(2^{\delta_* m'}l_p(w_\varphi)) \to l_1(l_q)$ be the identity operator. Then

$$e_n\left(\sum_{m=0}^{\infty}(\tilde{Q}_{n,m+1}-\tilde{Q}_{n,m}) : \hat{X}_p(\Omega)\to Y_q(\Omega)\right) =$$

$$= e_n(\overline{K}\circ \mathrm{id}\circ\overline{A} : \hat{X}_p(\Omega)\to Y_q(\Omega)) \underset{\mathfrak{Z}_0}{\overset{(38),(114),(115)}{\lesssim}}$$

$$\lesssim e_n(\mathrm{id} : l_\infty(2^{\delta_* m'}l_p(w_\varphi)) \to l_1(l_q)) =: E_n.$$

From Theorem C we get $E_n \underset{\mathfrak{Z}_0}{\lesssim} n^{\frac{1}{q}-\frac{1}{p}}$.

Applying Lemmas 1, 2, 3 together with (30), (33), (34), (109), we complete the proof of Theorem 1.

Remark 1. *Suppose that instead of Assumption 2 the following condition holds: for any $\xi \in \mathbf{V}(\mathcal{A})$ the set $\hat{F}(\xi)$ is the atom of* mes. *Then the assertion of Theorem 1 holds as well.*

3 Some particular cases

Let $(\mathcal{T}, \xi_0)$ be a tree, and let u, $w : \mathbf{V}(\mathcal{T}) \to \mathbb{R}_+$. We define the summation operator $S_{u,w,\mathcal{T}}$ by

$$S_{u,w,\mathcal{T}}f(\xi) = w(\xi)\sum_{\xi'\leqslant\xi}u(\xi')f(\xi'), \quad \xi\in\mathbf{V}(\mathcal{T}), \quad f:\mathbf{V}(\mathcal{T})\to\mathbb{R}.$$

Let $1 \leqslant p,\ q \leqslant \infty$. By $\mathfrak{S}^{p,q}_{\mathcal{T},u,w}$ we denote the minimal constant C in the inequality

$$\left(\sum_{\xi\in\mathbf{V}(\mathcal{T})}w^q(\xi)\left|\sum_{\xi'\leqslant\xi}u(\xi')f(\xi')\right|^q\right)^{1/q} \leqslant C\left(\sum_{\xi\in\mathbf{V}(\mathcal{T})}|f(\xi)|^p\right)^{1/p}, \quad f:\mathbf{V}(\mathcal{T})\to\mathbb{R}$$

(appropriately modified for $p=\infty$ or $q=\infty$).

Given a tree $(\mathcal{T}, \xi_0)$, $N \in \mathbb{Z}_+$, we denote by $[\mathcal{T}]_{\leqslant N}$ a tree with vertex set $\cup_{j=0}^N \mathbf{V}_j^{\mathcal{T}}(\xi_0)$.

Let $(\mathcal{A}, \xi_0)$ be a tree, let (1) hold, and let the measure space $(\Omega, \Sigma, \mathrm{mes})$, the partition Θ, the bijection $\hat F : \mathbf{V}(\mathcal{A}) \to \hat\Theta$ and the spaces $X_p(\Omega)$, $Y_q(\Omega)$, $\mathcal{P}(\Omega)$ be as defined at the page 3. Assumptions 1–3 will be replaced by the following conditions.

Assumption A. *There exist functions u, $w : \mathbf{V}(\mathcal{A}) \to (0, \infty)$ and a constant $c_2 \geqslant 1$ with the following property: for any vertex $\xi_* \in \mathbf{V}(\mathcal{A})$ there exists a linear continuous projection $P_{\xi_*} : Y_q(\Omega) \to \mathcal{P}(\Omega)$ such that for any vertex $\xi \geqslant \xi_*$ and for any function $f \in X_p(\Omega)$*

$$\|f - P_{\xi_*} f\|_{Y_q(\hat F(\xi))} \leqslant c_2 w(\xi) \sum_{\xi_* \leqslant \xi' \leqslant \xi} u(\xi') \|f\|_{X_p(\hat F(\xi'))}. \tag{116}$$

Assumption B. *There exists a number $\delta_* > 0$ such that for any vertex $\xi \in \mathbf{V}(\mathcal{A})$ and for any $n \in \mathbb{N}$, $m \in \mathbb{Z}_+$ there exists a partition $T_{m,n}(G)$ of the set $G = \hat F(\xi)$ with the following properties:*

1. $\operatorname{card} T_{m,n}(G) \leqslant c_2 \cdot 2^m n$.
2. *For any $E \in T_{m,n}(G)$ there exists a linear continuous operator $P_E : Y_q(\Omega) \to \mathcal{P}(E)$ such that for any function $f \in X_p(\Omega)$*
$$\|f - P_E f\|_{Y_q(E)} \leqslant c_2 (2^m n)^{-\delta_*} u(\xi) w(\xi) \|f\|_{X_p(E)}. \tag{117}$$
3. *For any $E \in T_{m,n}(G)$*
$$\operatorname{card}\{E' \in T_{m\pm 1,n}(G) :\, \mathrm{mes}(E \cap E') > 0\} \leqslant c_2. \tag{118}$$

Let $\mathbf{V}(\mathcal{A}) = \{\eta_{j,i}\}_{j \geqslant j_{\min}, i \in \tilde I_j}$, where $j_{\min} \geqslant 0$, $\tilde I_{j_{\min}} = \{1\}$. Suppose that $\eta_{j_{\min},1}$ is the minimal vertex of $\mathcal{A}$ and $\mathbf{V}^{\mathcal{A}}_{j-j_{\min}}(\eta_{j_{\min},1}) = \{\eta_{j,i}\}_{i \in \tilde I_j}$ for any $j \geqslant j_{\min}$.

Assumption C. *There exist numbers $\theta > 0$, $\gamma \in \mathbb{R}$, $\kappa \geqslant \frac{\theta}{q}$, α_u, $\alpha_w \in \mathbb{R}$, $c_3 \geqslant 1$, $m_* \in \mathbb{N}$ and an absolutely continuous function $\tau : (0, \infty) \to (0, \infty)$ such that $\lim\limits_{t \to +\infty} \frac{t\tau'(t)}{\tau(t)} = 0$ and the following conditions hold.*

1. *If $\kappa = \frac{\theta}{q}$, then $\alpha_w > \frac{1-\gamma}{q}$.*
2. *If $\kappa > \frac{\theta}{q}$, then $\alpha_u + \alpha_w = \frac{1}{p} - \frac{1}{q}$; if $\kappa = \frac{\theta}{q}$, then $\alpha_u + \alpha_w = \frac{1}{p}$.*
3. *For any $j' \geqslant j \geqslant j_{\min}$ and for any vertex $\xi \in \mathbf{V}^{\mathcal{A}}_{j-j_{\min}}(\eta_{j_{\min},1})$*
$$\operatorname{card} \mathbf{V}^{\mathcal{A}}_{j'-j}(\xi) \leqslant c_3 \cdot 2^{\theta m_*(j'-j)} \frac{j^\gamma \tau(m_* j)}{j'^\gamma \tau(m_* j')}. \tag{119}$$

4. For any $j \geqslant j_{\min}$, $i \in \tilde I_j$

$$u(\eta_{j,i}) = u_j = 2^{\kappa m_* j}(m_*j)^{-\alpha_u}, \quad w(\eta_{j,i}) = w_j = 2^{-\kappa m_* j}(m_*j)^{-\alpha_w}. \qquad (120)$$

5. Let $\kappa = \frac{\theta}{q}$. Then there exists a tree $\hat{\mathcal{A}}$ with the minimal vertex $\hat\zeta_0$, which satisfies the following conditions.

(a) For any $j' \geqslant j \geqslant j_{\min}$ and for any vertex $\xi \in \mathbf{V}^{\hat{\mathcal{A}}}_{j-j_{\min}}(\hat\zeta_0)$

$$c_3^{-1} \cdot 2^{\theta m_*(j'-j)} \frac{j^\gamma \tau(m_*j)}{j'^\gamma \tau(m_*j')} \leqslant \operatorname{card} \mathbf{V}^{\hat{\mathcal{A}}}_{j'-j}(\xi) \leqslant c_3 \cdot 2^{\theta m_*(j'-j)} \frac{j^\gamma \tau(m_*j)}{j'^\gamma \tau(m_*j')}. \qquad (121)$$

(b) Let $\{\overline{u}_j\}_{j\geqslant j_{\min}} \subset (0, \infty)$, $\{\overline{w}_j\}_{j\geqslant j_{\min}} \subset (0, \infty)$ be arbitrary sequences. Define the functions $\tilde u$, $\tilde w : \mathbf{V}(\mathcal{A}) \to (0, \infty)$, $\hat u$, $\hat w : \mathbf{V}(\hat{\mathcal{A}}) \to (0, \infty)$ by

$$\tilde u|_{\mathbf{V}^{\mathcal{A}}_{j-j_{\min}}(\eta_{j_{\min},1})} \equiv \overline{u}_j, \quad \hat u|_{\mathbf{V}^{\hat{\mathcal{A}}}_{j-j_{\min}}(\zeta_0)} \equiv \overline{u}_j,$$

$$\tilde w|_{\mathbf{V}^{\mathcal{A}}_{j-j_{\min}}(\eta_{j_{\min},1})} \equiv \overline{w}_j, \quad \hat w|_{\mathbf{V}^{\hat{\mathcal{A}}}_{j-j_{\min}}(\zeta_0)} \equiv \overline{w}_j, \quad j \geqslant j_{\min}.$$

Then for each $N \geqslant j \geqslant j_{\min}$, $i \in \tilde I_j$ there exists a vertex $\hat\xi \in \mathbf{V}^{\hat{\mathcal{A}}}_{j-j_{\min}}(\hat\zeta_0)$ such that

$$\mathfrak{S}^{p,q}_{[\mathcal{A}_{\eta_{j,i}}]_{\leqslant N-j}, \tilde u, \tilde w} \leqslant c_3 \mathfrak{S}^{p,q}_{[\hat{\mathcal{A}}_{\hat\xi}]_{\leqslant N-j}, \hat u, \hat w}.$$

Denote $\mathfrak{Z}_0 = (p,\, q,\, c_1,\, c_2,\, c_3,\, \theta,\, \gamma,\, \kappa,\, \alpha_u,\, \alpha_w,\, m_*,\, \delta_*,\, \tau)$.

Let $\xi_* \in \mathbf{V}(\mathcal{A})$, and let $\mathcal{D} \subset \mathcal{A}$ be a subtree with the minimal vertex ξ_*. Then from Assumption A it follows that

$$\|f - P_{\xi_*} f\|_{Y_q(\Omega_{\mathcal{D}})} \leqslant c_2 \mathfrak{S}^{p,q}_{\mathcal{D},u,w} \|f\|_{X_p(\Omega_{\mathcal{D}})}. \qquad (122)$$

From Theorem F in [36], Theorem 3.6 in [38] and Assumption C it follows that if $\xi_* \in \mathbf{V}^{\mathcal{A}}_{j-j_{\min}}(\xi_0)$, then in the case $\kappa > \frac{\theta}{q}$

$$\mathfrak{S}^{p,q}_{\mathcal{D},u,w} \underset{\mathfrak{Z}_0}{\lesssim} \sup_{s\geqslant j} u_s w_s \underset{\mathfrak{Z}_0}{\asymp} (m_*j)^{-\alpha_u-\alpha_w} = (m_*j)^{\frac{1}{q}-\frac{1}{p}}, \qquad (123)$$

and in the case $\kappa = \frac{\theta}{q}$

$$\mathfrak{S}^{p,q}_{\mathcal{D},u,w} \underset{\mathfrak{Z}_0}{\lesssim} \sup_{s\geqslant j} \left(\sum_{i=j}^{s} (m_*i)^{-p'\alpha_u} \cdot 2^{p'\kappa m_* i} \right)^{\frac{1}{p'}} \left(\sum_{i\geqslant s} (m_*i)^{-\alpha_w q} \cdot \tfrac{s^\gamma \tau(m_*s)}{i^\gamma \tau(m_*i)} \cdot 2^{-q\kappa m_* s} \right)^{\frac{1}{q}} \underset{\mathfrak{Z}_0}{\lesssim} \lesssim (m_*j)^{-\alpha_u-\alpha_w+\frac{1}{q}} = (m_*j)^{\frac{1}{q}-\frac{1}{p}}. \qquad (124)$$

Notice that the proof of (124) depends on condition 5 of Assumption C.

From Assumption B and conditions 2, 4 of Assumption C it follows that if $\xi \in \mathbf{V}^{\mathcal{A}}_{j-j_{\min}}(\xi_0)$, $G = \hat{F}(\xi)$, $E \in T_{m,n}(G)$, then

$$\begin{aligned} \|f - P_E f\|_{Y_q(E)} \leqslant c_2 \cdot (2^m n)^{-\delta_*} (m_* j)^{-\alpha_u - \alpha_w} \|f\|_{X_p(E)} \leqslant \\ \leqslant c_2 \cdot (2^m n)^{-\delta_*} (m_* j)^{\frac{1}{q} - \frac{1}{p}} \|f\|_{X_p(E)}. \end{aligned} \tag{125}$$

We construct the partition $\{\mathcal{A}_{t,i}\}_{t\geqslant t_0,\, i\in \hat{J}_t}$ as follows. Let

$$t_0 = \min\{t \in \mathbb{Z}_+ :\ 2^t > m_* j_{\min}\}.$$

Given $t \geqslant t_0$, we denote by Γ_t the maximal subgraph in $\mathcal{A}$ on the vertex set

$$\mathbf{V}(\Gamma_t) = \{\eta_{j,s} :\ 2^{t-1} \leqslant m_* j < 2^t,\ s \in \tilde{I}_j\}, \tag{126}$$

and by $\mathcal{A}_{t,i}$, $i \in \hat{J}_t$, the connected components of the graph Γ_t. By $\hat{\xi}_{t,i}$ we denote the minimal vertex of the tree $\mathcal{A}_{t,i}$. Then

$$\operatorname{card} \mathbf{V}(\Gamma_t) \underset{\mathfrak{Z}_0}{\overset{(119)}{\lesssim}} 2^{\theta \cdot 2^t} 2^{-\gamma t} \tau^{-1}(2^t). \tag{127}$$

Thus, Assumption 3 holds with $\overline{\nu}_t = 2^{\theta\cdot 2^t} 2^{-\gamma t}\tau^{-1}(2^t)$. From (125) we obtain that Assumption 2 holds.

Recall the notations $J_{t,\mathcal{D}} = \{i \in \hat{J}_t :\ \mathbf{V}(\mathcal{A}_{t,i}) \cap \mathbf{V}(\mathcal{D}) \neq \varnothing\}$, $\mathcal{D}_{t,i} = \mathcal{D} \cap \mathcal{A}_{t,i}$, where $\mathcal{D} \subset \mathcal{A}$ is a subtree.

In order to obtain Assumption 1, it is sufficient to prove the following assertion.

Lemma 7. *Let $\mathcal{D}$ be a subtree in $\mathcal{A}$ rooted at ξ_*. Then*

$$\|f - P_{\xi_*} f\|^q_{Y_q(\Omega_{\mathcal{D}})} \underset{\mathfrak{Z}_0}{\lesssim} \sum_{t=t_0}^{\infty} 2^{\left(1-\frac{q}{p}\right)t} \sum_{i\in J_{t,\mathcal{D}}} \|f\|^q_{X_p(\Omega_{\mathcal{D}_{t,i}})}. \tag{128}$$

Proof. By (116),

$$\|f - P_{\xi_*} f\|^q_{Y_q(\Omega_{\mathcal{D}})} \underset{\mathfrak{Z}_0}{\lesssim} \sum_{\xi\in\mathbf{V}(\mathcal{D})} w^q(\xi) \left(\sum_{\xi_*\leqslant\xi'\leqslant\xi} u(\xi') \|f\|_{X_p(\hat{F}(\xi'))}\right)^q.$$

Let $\kappa > \frac{\theta}{q}$. Then $\alpha_u + \alpha_w = \frac{1}{p} - \frac{1}{q}$ (see condition 2 of Assumption C). Repeating the proof of Lemma 5.1 in [41], we get that

$$\|f - P_{\xi_*} f\|^q_{Y_q(\Omega_{\mathcal{D}})} \underset{\mathfrak{Z}_0}{\lesssim} \sum_{\xi\in\mathbf{V}(\mathcal{D})} w^q(\xi) u^q(\xi) \|f\|^q_{X_p(\hat{F}(\xi))}.$$

If $\xi = \eta_{j,i}$, $2^{t-1} \leqslant m_* j < 2^t$, then $u^q(\xi) w^q(\xi) \overset{(120)}{=} (m_* j)^{-\frac{q}{p}+1} \underset{\mathfrak{Z}_0}{\asymp} 2^{\left(1-\frac{q}{p}\right)t}$. This together with the condition $p < q$ implies (128).

Now we consider the case $\kappa = \frac{\theta}{q}$.

Let $\xi_* \in \mathbf{V}(\mathcal{A}_{\hat{t},j_0})$, and let $\xi \in \mathbf{V}(\mathcal{D}_{\hat{t}+l,j_l})$. Then there exists a sequence $\{\hat{\xi}_{\hat{t}+s,j_s}\}_{s=1}^{l}$ such that $\xi_* < \hat{\xi}_{\hat{t}+1,j_1} < \hat{\xi}_{\hat{t}+2,j_2} < \dots < \hat{\xi}_{\hat{t}+l,j_l}$. Denote

$$\tilde{\xi}_{\hat{t},j_0} = \xi_*, \quad \tilde{\xi}_{\hat{t}+s,j_s} = \hat{\xi}_{\hat{t}+s,j_s}, 1 \leqslant s \leqslant l. \tag{129}$$

We have

$$\sum_{\xi_* \leqslant \xi' \leqslant \xi} u(\xi')f(\xi') = \sum_{s=0}^{l-1} \sum_{\tilde{\xi}_{\hat{t}+s,j_s} \leqslant \xi' < \tilde{\xi}_{\hat{t}+s+1,j_{s+1}}} u(\xi')\|f\|_{X_p(\hat{F}(\xi'))} + \sum_{\tilde{\xi}_{\hat{t}+l,j_l} \leqslant \xi' \leqslant \xi} u(\xi')\|f\|_{X_p(\hat{F}(\xi'))}. \tag{130}$$

Let $\tilde{\alpha}_w = \alpha_w - \frac{1}{q}$. By condition 2 of Assumption C,

$$\alpha_u + \tilde{\alpha}_w = \frac{1}{p} - \frac{1}{q}. \tag{131}$$

We have

$$\sum_{\xi \in \mathbf{V}(\mathcal{D}_{\hat{t}+l,\,j_l})} w^q(\xi) \underset{\mathfrak{Z}_0}{\overset{(119),(120),(126)}{\lesssim}}$$

$$\lesssim \sum_{2^{\hat{t}+l-1} \leqslant m_*j < 2^{\hat{t}+l}} 2^{-\theta m_* j}(m_*j)^{-\alpha_w q} \cdot 2^{\theta(m_*j - 2^{\hat{t}+l-1})} \left(\frac{2^{\hat{t}+l-1}}{m_*j}\right)^{\gamma} \frac{\tau(2^{\hat{t}+l-1})}{\tau(m_*j)} \underset{\mathfrak{Z}_0}{\overset{(41)}{\lesssim}}$$

$$\lesssim 2^{-\kappa q \cdot 2^{\hat{t}+l-1}} \cdot 2^{(\hat{t}+l)(-\alpha_w q+1)} = 2^{-\kappa q \cdot 2^{\hat{t}+l-1}} \cdot 2^{-(\hat{t}+l)\tilde{\alpha}_w q},$$

which implies

$$\|f - P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}_{\hat{t}+l,j_l}})} \underset{\mathfrak{Z}_0}{\overset{(116)}{\lesssim}} \sum_{\xi \in \mathbf{V}(\mathcal{D}_{\hat{t}+l,j_l})} w^q(\xi) \left(\sum_{\xi_* \leqslant \xi' \leqslant \xi} u(\xi')f(\xi')\right)^q \underset{\mathfrak{Z}_0}{\overset{(130)}{\lesssim}}$$

$$\lesssim \sum_{\xi \in \mathbf{V}(\mathcal{D}_{\hat{t}+l,j_l})} w^q(\xi) \left(\sum_{s=0}^{l-1} \sum_{\tilde{\xi}_{\hat{t}+s,j_s} \leqslant \xi' < \tilde{\xi}_{\hat{t}+s+1,j_{s+1}}} u(\xi')\|f\|_{X_p(\hat{F}(\xi'))}\right)^q +$$

$$+ \sum_{\xi \in \mathbf{V}(\mathcal{D}_{\hat{t}+l,j_l})} w^q(\xi) \left(\sum_{\tilde{\xi}_{\hat{t}+l,j_l} \leqslant \xi' \leqslant \xi} u(\xi')\|f\|_{X_p(\hat{F}(\xi'))}\right)^q \underset{\mathfrak{Z}_0}{\overset{(124),(126)}{\lesssim}}$$

$$\lesssim 2^{-q\kappa \cdot 2^{\hat{t}+l-1}} \cdot 2^{-q\tilde{\alpha}_w(\hat{t}+l)} \left(\sum_{s=0}^{l-1} \sum_{\tilde{\xi}_{\hat{t}+s,j_s} \leqslant \xi' < \tilde{\xi}_{\hat{t}+s+1,j_{s+1}}} u(\xi')\|f\|_{X_p(\hat{F}(\xi'))}\right)^q +$$

$$+2^{\left(1-\frac{q}{p}\right)(\hat{t}+l)}\|f\|^q_{X_p(\mathcal{D}_{\hat{t}+l,j_l})};$$

i.e.,

$$\|f-P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}_{\hat{t}+l,j_l}})} \underset{\mathfrak{Z}_0}{\lesssim} 2^{\left(1-\frac{q}{p}\right)(\hat{t}+l)}\|f\|^q_{X_p(\mathcal{D}_{\hat{t}+l,j_l})}+$$
$$+2^{-q\kappa\cdot 2^{\hat{t}+l-1}}\cdot 2^{-q\tilde{\alpha}_w(\hat{t}+l)}\left(\sum_{s=0}^{l-1}\sum_{\tilde{\xi}_{\hat{t}+s,j_s}\leqslant\xi'<\tilde{\xi}_{\hat{t}+s+1,j_{s+1}}} u(\xi')\|f\|_{X_p(\hat{F}(\xi'))}\right)^q. \qquad (132)$$

Denote

$$\{\zeta_{t,i}:\, i\in I'_t\}=\{\zeta\in\mathbf{V}_{\max}(\mathcal{D}\cap\Gamma_t):\, \mathbf{V}_1^{\mathcal{D}}(\zeta)\neq\varnothing\}. \qquad (133)$$

By (126),

$$\forall i\in I'_t\ \exists s\in\tilde{I}_{\lceil 2^t/m_*\rceil-1}:\quad \zeta_{t,i}=\eta_{\lceil 2^t/m_*\rceil-1,s}. \qquad (134)$$

Let us define the tree $\hat{\mathcal{D}}$ with vertex set $\{\xi_*\}\cup\left(\cup_{t\geqslant\hat{t}}\{\zeta_{t,i}:\, i\in I'_t\}\right)$. The partial order on $\mathbf{V}(\hat{\mathcal{D}})$ is defined as follows. We set

$$\mathbf{V}_1^{\hat{\mathcal{D}}}(\zeta_{t,i})=\{\zeta_{t+1,j}:\, \zeta_{t+1,j}\in\mathcal{D}_{\zeta_{t,i}}\}.$$

If $\xi_*\notin\{\zeta_{\hat{t},i}:\, i\in I'_{\hat{t}}\}$, then we set $\mathbf{V}_1^{\hat{\mathcal{D}}}(\xi_*)=\{\zeta_{\hat{t},i}:\, i\in I'_{\hat{t}}\}$.

Further, we define the functions φ, $\hat{u}$, $\hat{w}:\mathbf{V}(\hat{\mathcal{D}})\to\mathbb{R}_+$. We set

$$\varphi(\zeta_{t,i})=\sum_{\xi'\in\mathbf{V}(\Gamma_t)\cap\mathbf{V}(\mathcal{D}),\,\xi'\leqslant\zeta_{t,i}}\frac{u(\xi')}{u(\zeta_{t,i})}\|f\|_{X_p(\hat{F}(\xi'))}, \qquad (135)$$

$$\hat{w}(\zeta_{t,i})=2^{-\kappa\cdot 2^t}\cdot 2^{-\tilde{\alpha}_w t},\quad \hat{u}(\zeta_{t,i})=2^{\kappa\cdot 2^t}\cdot 2^{-\alpha_u t}\overset{(120),(134)}{\underset{\mathfrak{Z}_0}{\asymp}} u(\zeta_{t,i}). \qquad (136)$$

If $\xi_*\notin\{\zeta_{\hat{t},i}:\, i\in I'_{\hat{t}}\}$, then we set $\varphi(\xi_*)=\hat{w}(\xi_*)=\hat{u}(\xi_*)=0$.

Let $t\geqslant\hat{t}$, $j\in J_{t,\mathcal{D}}$. If $t>\hat{t}$, then $\hat{\xi}_{t,j}\in\mathbf{V}(\mathcal{D})$. We take $i\in I'_{t-1}$ such that $\hat{\xi}_{t,j}\in\mathbf{V}_1^{\mathcal{D}}(\zeta_{t-1,i})$. Then

$$\|f-P_{\xi_*}f\|^q_{Y_q(\mathcal{D}_{t,j})}\overset{(132),(136)}{\underset{\mathfrak{Z}_0}{\lesssim}} 2^{\left(1-\frac{q}{p}\right)t}\|f\|^q_{X_p(\mathcal{D}_{t,j})}+\hat{w}^q(\zeta_{t-1,i})\left(\sum_{\zeta'\in\mathbf{V}(\hat{\mathcal{D}}),\,\zeta'\leqslant\zeta_{t-1,i}}\hat{u}(\zeta')\varphi(\zeta')\right)^q. \qquad (137)$$

Notice that $\operatorname{card}\mathbf{V}_1^{\mathcal{D}}(\zeta_{t,i})\overset{(119)}{\underset{\mathfrak{Z}_0}{\lesssim}} 1$. Summing (137) over all $t\geqslant\hat{t}$, $j\in J_{t,\mathcal{D}}$, we obtain

$$\|f-P_{\xi_*}f\|^q_{Y_q(\Omega_{\mathcal{D}})}\underset{\mathfrak{Z}_0}{\lesssim}\sum_{t=\hat{t}}^{\infty}2^{\left(1-\frac{q}{p}\right)t}\sum_{j\in J_{t,\mathcal{D}}}\|f\|^q_{X_p(\Omega_{\mathcal{D}_{t,j}})}+$$
$$+\sum_{t=\hat{t}}^{\infty}\sum_{i\in I'_t}\hat{w}^q(\zeta_{t,i})\left(\sum_{\zeta'\in\mathbf{V}(\hat{\mathcal{D}}),\,\zeta'\leqslant\zeta_{t,i}}\hat{u}(\zeta')\varphi(\zeta')\right)^q. \qquad (138)$$

Let us estimate the second summand (we denote it by L).

Let $\hat u(\zeta)=\hat u_1(\zeta)\hat u_2(\zeta)$, $\hat u_1(\zeta_{t,i})=2^{\varepsilon t}$, where $\varepsilon=\varepsilon(\mathfrak{Z}_0)>0$ is sufficiently small (it will be chosen later); if $\xi_*\notin\{\zeta_{\hat t,i}:\ i\in I'_{\hat t}\}$, then we set $\hat u_1(\xi_*)=\hat u_2(\xi_*)=0$. By Hölder's inequality,

$$L=\sum_{\zeta\in\mathbf{V}(\hat{\mathcal D})}\hat w^q(\zeta)\left(\sum_{\zeta'\leqslant\zeta}\hat u_1(\zeta')\hat u_2(\zeta')\varphi(\zeta')\right)^q\leqslant$$

$$\leqslant\sum_{\zeta\in\mathbf{V}(\hat{\mathcal D})}\hat w^q(\zeta)\left(\sum_{\zeta'\leqslant\zeta}\hat u_1^{q'}(\zeta')\right)^{\frac{q}{q'}}\sum_{\zeta'\leqslant\zeta}\hat u_2^q(\zeta')\varphi^q(\zeta')\underset{\mathfrak{Z}_0}{\lesssim}$$

$$\lesssim\sum_{\zeta\in\mathbf{V}(\hat{\mathcal D})}\hat w^q(\zeta)\hat u_1^q(\zeta)\sum_{\zeta'\leqslant\zeta}\hat u_2^q(\zeta')\varphi^q(\zeta')=\sum_{\zeta'\in\mathbf{V}(\hat{\mathcal D})}\hat u_2^q(\zeta')\varphi^q(\zeta')\sum_{\zeta\geqslant\zeta'}\hat w^q(\zeta)\hat u_1^q(\zeta)=:M.$$

By (134), there exists $s\in\tilde I_{\lceil 2^t/m_*\rceil-1}$ such that

$$\operatorname{card}\mathbf{V}_l^{\hat{\mathcal D}}(\zeta_{t,i})=\operatorname{card}\{i'\in I'_{t+l}:\ \zeta_{t+l,i'}\geqslant\zeta_{t,i}\}\leqslant$$

$$\leqslant\operatorname{card}\mathbf{V}^{\mathcal A}_{\lceil 2^{t+l}/m_*\rceil-\lceil 2^t/m_*\rceil}(\eta_{\lceil 2^t/m_*\rceil-1,s})\underset{\mathfrak{Z}_0}{\overset{(119)}{\lesssim}}\frac{2^{\theta\cdot 2^{t+l}}2^{-\gamma(t+l)}\tau^{-1}(2^{t+l})}{2^{\theta\cdot 2^t}2^{-\gamma t}\tau^{-1}(2^t)}.$$

This together with relations $\kappa=\frac{\theta}{q}$, $\alpha_w>\frac{1-\gamma}{q}$ (see condition 1 of Assumption C) yields that for sufficiently smalll $\varepsilon>0$

$$\sum_{\zeta\geqslant\zeta_{t,i}}\hat w^q(\zeta)\hat u_1^q(\zeta)\underset{\mathfrak{Z}_0}{\overset{(136)}{\lesssim}}\sum_{l\geqslant 0}2^{-q\kappa\cdot 2^{t+l}}\cdot 2^{-q\tilde\alpha_w(t+l)}\cdot 2^{q\varepsilon(t+l)}\frac{2^{\theta\cdot 2^{t+l}}2^{-\gamma(t+l)}\tau^{-1}(2^{t+l})}{2^{\theta\cdot 2^t}2^{-\gamma t}\tau^{-1}(2^t)}=$$

$$=2^{-\theta\cdot 2^t}\cdot 2^{\gamma t}\sum_{l\geqslant 0}2^{-q\left(\tilde\alpha_w+\frac{\gamma}{q}-\varepsilon\right)(t+l)}\frac{\tau(2^t)}{\tau(2^{t+l})}\underset{\mathfrak{Z}_0}{\overset{(41)}{\lesssim}}2^{-q\kappa\cdot 2^t}\cdot 2^{-q(\tilde\alpha_w-\varepsilon)t}\overset{(136)}{=}\hat w^q(\zeta_{t,i})\hat u_1^q(\zeta_{t,i}).$$

Thus,

$$M\underset{\mathfrak{Z}_0}{\lesssim}\sum_{\zeta'\in\mathbf{V}(\hat{\mathcal D})}\hat w^q(\zeta')\hat u_1^q(\zeta')\hat u_2^q(\zeta')\varphi^q(\zeta')=\sum_{\zeta'\in\mathbf{V}(\hat{\mathcal D})}\hat w^q(\zeta')\hat u^q(\zeta')\varphi^q(\zeta')\overset{(131),(136)}{=}$$

$$=\sum_{t=\hat t}^{\infty}\sum_{j\in J_{t,\mathcal D}}2^{\left(1-\frac{q}{p}\right)t}\sum_{i\in I'_t:\,\zeta_{t,i}\in\mathbf{V}(\mathcal D_{t,j})}\varphi^q(\zeta_{t,i});$$

i.e.,

$$\sum_{t=\hat t}^{\infty}\sum_{i\in I'_t}\hat w^q(\zeta_{t,i})\left(\sum_{\zeta'\in\mathbf{V}(\hat{\mathcal D}),\,\zeta'\leqslant\zeta_{t,i}}\hat u(\zeta')\varphi(\zeta')\right)^q\underset{\mathfrak{Z}_0}{\lesssim}\sum_{t=\hat t}^{\infty}\sum_{j\in J_{t,\mathcal D}}2^{\left(1-\frac{q}{p}\right)t}\sum_{i\in I'_t:\,\zeta_{t,i}\in\mathbf{V}(\mathcal D_{t,j})}\varphi^q(\zeta_{t,i}).\tag{139}$$

It remains to prove that for any $t \geqslant \hat{t}$, $j \in J_{t,\mathcal{D}}$

$$\sum_{i\in I'_t\colon \zeta_{t,i}\in \mathbf{V}(\mathcal{D}_{t,j})} \varphi^q(\zeta_{t,i}) \underset{\mathfrak{Z}_0}{\lesssim} \|f\|^q_{X_p(\Omega_{\mathcal{D}_{t,j}})}. \tag{140}$$

Then (138), (139), (140) imply (128).

By (129) and (135),

$$\sum_{i\in I'_t\colon \zeta_{t,i}\in \mathbf{V}(\mathcal{D}_{t,j})} \varphi^q(\zeta_{t,i}) = \sum_{i\in I'_t\colon \zeta_{t,i}\in \mathbf{V}(\mathcal{D}_{t,j})} \left(\sum_{\tilde{\xi}_{t,j}\leqslant \xi'\leqslant \zeta_{t,i}} \frac{u(\xi')}{u(\zeta_{t,i})}\|f\|_{X_p(\hat{F}(\xi'))}\right)^q =$$

$$= \sum_{i\in I'_t\colon \zeta_{t,i}\in \mathbf{V}(\mathcal{D}_{t,j})} u^{-q}(\zeta_{t,i}) \left(\sum_{\tilde{\xi}_{t,j}\leqslant \xi'\leqslant \zeta_{t,i}} u(\xi')\|f\|_{X_p(\hat{F}(\xi'))}\right)^q =: S.$$

From (133) it follows that $\zeta_{t,i} \in \mathbf{V}_{\max}(\mathcal{A}_{t,j})$.

We denote $\mathcal{A}'_{t,j} = (\mathcal{A}_{t,j})_{\tilde{\xi}_{t,j}}$. Then $\mathbf{V}(\mathcal{D}_{t,j}) \subset \mathbf{V}(\mathcal{A}'_{t,j})$. Given $\sigma \geqslant 0$, we set

$$w_{(\sigma)}(\xi) = \left\{\begin{array}{l} u^{-1}(\xi) \quad \text{for} \quad \xi \in \mathbf{V}_{\max}(\mathcal{A}'_{t,j}), \\ \sigma \quad \text{for} \quad \xi \in \mathbf{V}(\mathcal{A}'_{t,j})\backslash \mathbf{V}_{\max}(\mathcal{A}'_{t,j}). \end{array}\right.$$

Then

$$S \leqslant \left[\mathfrak{S}^{p,q}_{\mathcal{A}'_{t,j},u,w_{(\sigma)}}\right]^q \|f\|_{X_p(\Omega_{\mathcal{D}_{t,j}})}.$$

We claim that for sufficiently small $\sigma > 0$ the estimate $\mathfrak{S}^{p,q}_{\mathcal{A}'_{t,j},u,w_{(\sigma)}} \underset{\mathfrak{Z}_0}{\lesssim} 1$ holds. This implies (140).

Let $\tilde{\xi}_{t,j} \in \mathbf{V}^{\mathcal{A}}_{s_*-j_{\min}}(\eta_{j_{\min},1})$. From (126) and (129) it follows that

$$2^{t-1} \leqslant m_* s_* < 2^t. \tag{141}$$

Moreover, $\mathcal{A}'_{t,j} = [\mathcal{A}_{\tilde{\xi}_{t,j}}]_{\leqslant \lceil 2^t/m_*\rceil -1-s_*}$ (since $\hat{\xi}_{t+1,i} \in \mathbf{V}^{\mathcal{A}}_{\lceil 2^t/m_*\rceil - j_{\min}}(\eta_{j_{\min},1})$, $i \in \hat{J}_t$). By condition 5b of Assumption C, there exists a vertex $\hat{\zeta} \in \mathbf{V}^{\hat{\mathcal{A}}}_{s_*-j_{\min}}(\hat{\zeta}_0)$ such that for the tree $\hat{\mathcal{A}}_{t,j} := [\hat{\mathcal{A}}_{\hat{\zeta}}]_{\leqslant \lceil 2^t/m_*\rceil -1-s_*}$ and for the functions $\overline{u}$, $\overline{w}_{(\sigma)} : \mathbf{V}(\hat{\mathcal{A}}_{t,j}) \to (0, \infty)$ defined by

$$\overline{u}(\xi) = u_s, \quad \xi \in \mathbf{V}^{\hat{\mathcal{A}}}_{s-s_*}(\hat{\zeta}_0), \quad \overline{w}(\xi) = \left\{\begin{array}{l} u^{-1}_{\lceil 2^t/m_*\rceil -1}, \quad \xi \in \mathbf{V}^{\hat{\mathcal{A}}}_{\lceil 2^t/m_*\rceil -1-s_*}(\hat{\zeta}_0), \\ \sigma, \quad \xi \in \mathbf{V}^{\hat{\mathcal{A}}}_{s-s_*}(\hat{\zeta}_0), \quad s < \lceil 2^t/m_*\rceil - 1, \end{array}\right. \tag{142}$$

the inequality $\mathfrak{S}^{p,q}_{\mathcal{A}'_{t,j},u,w_{(\sigma)}} \underset{\mathfrak{Z}_0}{\lesssim} \mathfrak{S}^{p,q}_{\hat{\mathcal{A}}_{t,j},\overline{u},\overline{w}_{(\sigma)}}$ holds.

From (121) it follows that we can apply Theorem 3.6 in [38] and estimate $\mathfrak{S}^{p,q}_{\hat{\mathcal{A}}_{t,j},\overline{u},\overline{w}_{(\sigma)}}$ from above. For sufficiently small $\sigma = \sigma(\mathfrak{Z}_0, t) > 0$ we get

$$\mathfrak{S}^{p,q}_{\hat{\mathcal{A}}_{t,j},\overline{u},\overline{w}_{(\sigma)}} \overset{(121),(142)}{\underset{\mathfrak{Z}_0}{\lesssim}} \sup_{s_*\leqslant s<\lceil 2^t/m_*\rceil} \left(\sum_{l=s_*}^{s} u_l^{p'}\right)^{1/p'} \times$$

$$\times\left(\sum_{l=s}^{\lceil 2^t/m_*\rceil-2}\sigma^q\cdot 2^{\theta m_*(l-s)}\frac{s^\gamma\tau(m_*s)}{l^\gamma\tau(m_*l)}+u^{-q}_{\lceil 2^t/m_*\rceil-1}\cdot 2^{\theta(2^t-m_*s)}\frac{(m_*s)^\gamma\tau(m_*s)}{2^{\gamma t}\tau(2^t)}\right)^{1/q}\underset{\mathfrak{Z}_0}{\overset{(120),(141)}{\lesssim}}$$

$$\lesssim\sup_{s_*\leqslant s\leqslant\lceil 2^t/m_*\rceil-1}\left(\sum_{l=s_*}^{s}2^{p'\kappa m_*l}\cdot(m_*l)^{-\alpha_u p'}\right)^{1/p'}\left(2^{-q\kappa\cdot 2^t}\cdot 2^{q\alpha_u t}\cdot 2^{\theta(2^t-m_*s)}\right)^{1/q}.$$

Since $\theta=q\kappa$, we have

$$\mathfrak{S}^{p,q}_{\hat{\mathcal{A}}_{t,j},\overline{u},\overline{w}(\sigma)}\underset{\mathfrak{Z}_0}{\lesssim}\sup_{s_*\leqslant s\leqslant\lceil 2^t/m_*\rceil-1}2^{\kappa m_*s}\cdot(m_*s)^{-\alpha_u}\cdot 2^{-\kappa m_*s}\cdot 2^{\alpha_u t}\underset{\mathfrak{Z}_0}{\overset{(141)}{\lesssim}}1.$$

This completes the proof. □

Thus, Assumptions 1, 2, 3 hold, and by Theorem 1 we get (12).

Now we suppose that instead of Assumption C the following condition holds.

Assumption D. *There exist numbers $m_*\in\mathbb{N}$, $\kappa>0$, $\gamma\leqslant 0$, $\nu\in\mathbb{R}$, $\alpha_*\in\mathbb{R}$, $\lambda_u\in\mathbb{R}$, $\lambda_w\in\mathbb{R}$ such that $\lambda_u+\lambda_w=\frac{1}{p}-\frac{1}{q}$ and the following assertions hold.*

1. *For any $j'\geqslant j\geqslant j_{\min}$ and for any vertex $\xi\in\mathbf{V}^{\mathcal{A}}_{j-j_{\min}}(\xi_0)$*

$$\operatorname{card}\mathbf{V}^{\mathcal{A}}_{j'-j}(\xi)\leqslant c_3\frac{(m_*j)^\gamma|\log(m_*j)|^\nu}{(m_*j')^\gamma|\log(m_*j')|^\nu}. \tag{143}$$

2. *For any $j\geqslant j_{\min}$, $i\in\tilde{I}_j$*

$$\begin{aligned}u(\eta_{j,i})=u_j=2^{\kappa m_*j}(m_*j)^{\alpha_*}|\log(m_*j)|^{-\lambda_u},\\ w(\eta_{j,i})=w_j=2^{-\kappa m_*j}(m_*j)^{-\alpha_*}|\log(m_*j)|^{-\lambda_w}.\end{aligned} \tag{144}$$

The partition $\{\mathcal{A}_{t,i}\}_{t\geqslant t_0,\,i\in\hat{J}_t}$ is defined as follows. Let

$$t_0=\min\{t\in\mathbb{Z}_+:\ 2^{2^t}>m_*j_{\min}\}.$$

Given $t\geqslant t_0$, we denote by Γ_t the maximal subgraph in $\mathcal{A}$ on the vertex set $\{\eta_{j,s}:\ 2^{2^{t-1}}\leqslant m_*j<2^{2^t},\ s\in\tilde{I}_j\}$, and by $\mathcal{A}_{t,i}$, $i\in\hat{J}_t$, the connected components of Γ_t. Then

$$\operatorname{card}\mathbf{V}(\Gamma_t)\underset{\mathfrak{Z}_0}{\lesssim}2^{(1-\gamma)2^t}2^{-\nu t}. \tag{145}$$

Repeating the proof of Lemma 5.1 in [41] and taking into account that $p<q$, we get the following assertion.

Lemma 8. *Let $\mathcal{D}$ be a subtree in $\mathcal{A}$, and let ξ_* be its minimal vertex. Then*

$$\|f-P_{\xi_*}f\|^q_{Y_q(\mathcal{D})}\underset{\mathfrak{Z}_0}{\lesssim}\sum_{t=t_0}^{\infty}2^{(1-\frac{q}{p})t}\sum_{i\in J_{t,\mathcal{D}}}\|f\|^q_{X_p(\Omega_{\mathcal{D}_{t,i}})}. \tag{146}$$

From Assumption B, (145) and (146) we obtain Assumptions 1, 2, 3.

Remark 2. *Suppose that Assumption B is replaced by the following condition: for any $\xi \in \mathbf{V}(\mathcal{A})$ the set $\hat{F}(\xi)$ is the atom of* mes. *Then the assertion of Theorem 1 holds as well (see Remark 1).*

4 Estimates for entropy numbers of embeddings of weighted Sobolev spaces

Let us define the weighted Sobolev class $W^r_{p,g}(\Omega)$ and the weighted Lebesgue space $L_{q,v}(\Omega)$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $g,\, v : \Omega \to (0,\, \infty)$ be measurable functions. For each measurable vector-valued function $\psi :\; \Omega \to \mathbb{R}^l$, $\psi = (\psi_k)_{1\leqslant k\leqslant l}$, and for each $p \in [1,\, \infty)$, we put

$$\|\psi\|_{L_p(\Omega)} = \left\| \max_{1\leqslant k\leqslant l} |\psi_k| \right\|_p = \left(\int\limits_\Omega \max_{1\leqslant k\leqslant l} |\psi_k(x)|^p \, dx \right)^{1/p}.$$

Let $\overline{\beta} = (\beta_1,\, \dots,\, \beta_d) \in \mathbb{Z}^d_+ := (\mathbb{N} \cup \{0\})^d$, $|\overline{\beta}| = \beta_1 + \ldots + \beta_d$. For any distribution f defined on Ω we write $\nabla^r f = \left(\partial^r f/\partial x^{\overline{\beta}}\right)_{|\overline{\beta}|=r}$ (here partial derivatives are taken in the sense of distributions), and denote by $l_{r,d}$ the number of components of the vector-valued distribution $\nabla^r f$. We set

$$W^r_{p,g}(\Omega) = \left\{ f :\; \Omega \to \mathbb{R} \middle| \; \exists \psi :\; \Omega \to \mathbb{R}^{l_{r,d}} :\; \|\psi\|_{L_p(\Omega)} \leqslant 1,\, \nabla^r f = g \cdot \psi \right\}$$

$\Big($we denote the corresponding function ψ by $\dfrac{\nabla^r f}{g}\Big)$,

$$\|f\|_{L_{q,v}(\Omega)} = \|f\|_{q,v} = \|fv\|_{L_q(\Omega)}, \qquad L_{q,v}(\Omega) = \{f : \Omega \to \mathbb{R} |\;\; \|f\|_{q,v} < \infty\}.$$

We call the set $W^r_{p,g}(\Omega)$ a weighted Sobolev class. Observe that if $g \in L^{\rm loc}_{p'}(\Omega)$, then $\nabla^r f \in L^{\rm loc}_1(\Omega)$.

For $x \in \mathbb{R}^d$ and $a > 0$ we shall denote by $B_a(x)$ the closed Euclidean ball of radius a in $\mathbb{R}^d$ centered at the point x.

Definition 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $a > 0$. We say that $\Omega \in \mathbf{FC}(a)$ if there exists a point $x_* \in \Omega$ such that, for any $x \in \Omega$, there exist a number $T(x) > 0$ and a curve $\gamma_x : [0,\, T(x)] \to \Omega$ with the following properties:*

1. *$\gamma_x \in AC[0,\, T(x)]$, $\left|\frac{d\gamma_x(t)}{dt}\right| = 1$ a.e.,*
2. *$\gamma_x(0) = x$, $\gamma_x(T(x)) = x_*$,*

3. $B_{at}(\gamma_x(t)) \subset \Omega$ *for any* $t \in [0, T(x)]$.

Definition 3. *We say that* Ω *satisfies the John condition (and call* Ω *a John domain) if* $\Omega \in \mathbf{FC}(a)$ *for some* $a > 0$.

For a bounded domain the John condition is equivalent to the flexible cone condition (see the definition in [2]). As examples of such domains we can take

1. domains with Lipschitz boundary;
2. the Koch's snowflake;
3. domains $\Omega = \cup_{0<t\leqslant T} \mathrm{int}\, B_{ct}(\gamma(t))$, where $\gamma : [0, T] \to \mathbb{R}^d$ is a curve with natural parametrization and $c > 0$.

Domains with zero inner angles do not satisfy the John condition.

We denote by $\mathbb{H}$ the set of all nondecreasing positive functions defined on $(0, 1]$.

Definition 4. (see [5]). *Let* $\Gamma \subset \mathbb{R}^d$ *be a nonempty compact set and* $h \in \mathbb{H}$. *We say that* Γ *is an* h*-set if there are a constant* $c_* \geqslant 1$ *and a finite countably additive measure* μ *on* $\mathbb{R}^d$ *such that* $\operatorname{supp}\mu = \Gamma$ *and*

$$c_*^{-1} h(t) \leqslant \mu(B_t(x)) \leqslant c_* h(t) \tag{147}$$

for any $x \in \Gamma$ *and* $t \in (0, 1]$.

Example 1. *Let* $\Gamma \subset \mathbb{R}^d$ *be a Lipschitz manifold of dimension* k*,* $0 \leqslant k < d$. *Then* Γ *is an* h*-set with* $h(t) = t^k$.

Example 2. *Let* $\Gamma \subset \mathbb{R}^2$ *be the Koch curve. Then* Γ *is an* h*-set with* $h(t) = t^{\log 4/\log 3}$ *(see [30, p. 66–68]).*

Let $\Omega \in \mathbf{FC}(a)$ be a bounded domain, and let $\Gamma \subset \partial\Omega$ be an h-set. Below we consider a function $h \in \mathbb{H}$ which has the following form near zero:

$$h(t) = t^\theta |\log t|^\gamma \tau(|\log t|), \quad 0 \leqslant \theta < d, \tag{148}$$

where $\tau : (0, +\infty) \to (0, +\infty)$ is an absolutely continuous function such that

$$\frac{t\tau'(t)}{\tau(t)} \underset{t\to+\infty}{\to} 0. \tag{149}$$

Let $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, $r \in \mathbb{N}$, $\delta := r + \frac{d}{q} - \frac{d}{p} > 0$, $\beta_g, \beta_v \in \mathbb{R}$, $g(x) = \varphi_g(\mathrm{dist}_{|\cdot|}(x, \Gamma))$, $v(x) = \varphi_v(\mathrm{dist}_{|\cdot|}(x, \Gamma))$,

$$\varphi_g(t) = t^{-\beta_g} |\log t|^{-\alpha_g} \rho_g(|\log t|), \quad \varphi_v(t) = t^{-\beta_v} |\log t|^{-\alpha_v} \rho_v(|\log t|), \tag{150}$$

where ρ_g and ρ_v are absolutely continuous functions such that

$$\frac{t\rho_g'(t)}{\rho_g(t)} \underset{t\to+\infty}{\to} 0, \quad \frac{t\rho_v'(t)}{\rho_v(t)} \underset{t\to+\infty}{\to} 0. \tag{151}$$

Moreover, assume that

$$\beta_v < \frac{d-\theta}{q} \quad \text{or} \quad \beta_v = \frac{d-\theta}{q}, \quad \alpha_v > \frac{1-\gamma}{q}. \tag{152}$$

Without loss of generality we may consider $\overline{\Omega} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$.

Denote

$$\beta = \beta_g + \beta_v, \quad \alpha = \alpha_g + \alpha_v, \quad \rho(y) = \rho_g(y)\rho_v(y),$$

$\mathfrak{Z} = (r, d, p, q, g, v, h, a, c_*)$, $\mathfrak{Z}_* = (\mathfrak{Z}, R)$, where c_* is the constant from Definition 4 and $R = \operatorname{diam}\Omega$.

Denote by $\mathcal{P}_{r-1}(\mathbb{R}^d)$ the space of polynomials on $\mathbb{R}^d$ of degree not exceeding $r-1$. For a measurable set $E \subset \mathbb{R}^d$ we put

$$\mathcal{P}_{r-1}(E) = \{f|_E : f \in \mathcal{P}_{r-1}(\mathbb{R}^d)\}.$$

Observe that $W^r_{p,g}(\Omega) \supset \mathcal{P}_{r-1}(\Omega)$.

In Theorems 2, 3 the conditions on weights are such that $W^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and there exist $M > 0$ and a linear continuous operator $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any function $f \in W^r_{p,g}(\Omega)$

$$\|f - Pf\|_{L_{q,v}(\Omega)} \leqslant M \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)} \tag{153}$$

(see [40–43]).

Denote $\mathcal{W}^r_{p,g}(\Omega) = \operatorname{span} W^r_{p,g}(\Omega)$, $\hat{\mathcal{W}}^r_{p,g}(\Omega) = \{f - Pf : \ f \in \mathcal{W}^r_{p,g}(\Omega)\}$. Let $\hat{\mathcal{W}}^r_{p,g}(\Omega)$ be equipped with norm $\|f\|_{\hat{\mathcal{W}}^r_{p,g}(\Omega)} := \left\|\frac{\nabla^r f}{g}\right\|_{L_p(\Omega)}$. Denote by $I : \hat{\mathcal{W}}^r_{p,g}(\Omega) \to L_{q,v}(\Omega)$ the embedding operator. From (153) it follows that it is continuous.

First we consider the case $0 < \theta < d$. We set

$$\alpha_0 := \begin{cases} \alpha & \text{for} \quad \beta_v < \frac{d-\theta}{q}, \\ \alpha - \frac{1}{q} & \text{for} \quad \beta_v = \frac{d-\theta}{q}. \end{cases}$$

In [37] the estimates for entropy numbers $e_n(I : \hat{\mathcal{W}}^r_{p,g}(\Omega) \to L_{q,v}(\Omega))$ were obtained under the following conditions. In the case $\delta - \beta > \theta\left(\frac{1}{q} - \frac{1}{p}\right)_+$ we assumed that $\frac{\delta}{d} \neq \frac{\delta-\beta}{\theta}$. In the case $\delta - \beta = \theta\left(\frac{1}{q} - \frac{1}{p}\right)_+$ we assumed that $\alpha_0 \neq \frac{1}{p} - \frac{1}{q}$ for $p < q$. Now we obtain estimates for $p < q$, $\beta - \delta = 0$, $\alpha_0 = \frac{1}{p} - \frac{1}{q}$.

Theorem 2. *Suppose that the conditions (148)–(152) hold, $\rho_g \equiv 1$, $\rho_v \equiv 1$. Let $0 < \theta < d$, $p < q$, $\beta = \delta$, and let $\alpha_0 = \frac{1}{p} - \frac{1}{q}$. Then*

$$e_n(I : \hat{\mathcal{W}}^r_{p,g}(\Omega) \to L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{\frac{1}{q}-\frac{1}{p}}.$$

Now we consider the case $\theta = 0$, $\beta_v < \frac{d}{q}$. We assume that

$$\rho_g(t) = |\log t|^{-\lambda_g}, \quad \rho_v = |\log t|^{-\lambda_v}, \quad \tau(t) = |\log t|^{\nu}. \tag{154}$$

Denote $\lambda = \lambda_g + \lambda_v$.

In [37] estimates of entropy numbers were obtained in the following cases:

1. $\delta - \beta > \theta \left(\frac{1}{q} - \frac{1}{p}\right)_+$,
2. $\delta - \beta = \theta \left(\frac{1}{q} - \frac{1}{p}\right)_+$, $\alpha > (1-\gamma)\left(\frac{1}{q} - \frac{1}{p}\right)_+$, $\frac{\delta}{d} \ne \frac{\alpha}{1-\gamma}$,
3. $\delta - \beta = \theta \left(\frac{1}{q} - \frac{1}{p}\right)_+$, $\alpha = (1-\gamma)\left(\frac{1}{q} - \frac{1}{p}\right)_+$, and if $p < q$, then $\lambda \ne \frac{1}{p} - \frac{1}{q}$.

Here we obtain the estimates for $p < q$, $\delta - \beta = 0$, $\alpha = 0$, $\lambda = \frac{1}{p} - \frac{1}{q}$.

Theorem 3. *Let the conditions (148), (150), (154) hold, and let $\theta = 0$, $\beta - \delta = 0$, $\beta_v < \frac{d}{q}$, $\alpha = 0$, $p < q$, $\lambda = \frac{1}{p} - \frac{1}{q}$. Then*

$$e_n(I : \hat{\mathcal{W}}^r_{p,g}(\Omega) \to L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{\frac{1}{q}-\frac{1}{p}}.$$

Proof of Theorems 2 and 3. The lower estimates are proved similarly as in [37].

Let us obtain upper estimates. To this end, we apply Theorem 1. Consider the tree $\mathcal{A}$ with vertex set $\{\eta_{j,i}\}_{j\geqslant j_{\min}, i\in \tilde{I}_j}$ and the partition of the domain Ω into subdomains $\Omega[\eta_{j,i}]$, as defined in [40], [41]. We set $\hat{F}(\eta_{j,i}) = \Omega[\eta_{j,i}]$. Let the number $\overline{s} = \overline{s}(a, d) \geqslant 4$ be as defined in [40]. Repeating arguments from the proof of Theorem 2 in [41] (without summation over vertices $\xi \in \mathbf{V}(\mathcal{A}_{\xi_*})$), we obtain that Assumption A holds with $u(\eta_{j,i}) = \varphi_g(2^{-\overline{s}j}) \cdot 2^{-\left(r-\frac{d}{p}\right)\overline{s}j}$, $w(\eta_{j,i}) = \varphi_v(2^{-\overline{s}j}) \cdot 2^{-\frac{d\overline{s}j}{q}}$. Assumption B holds with $\delta_* = \frac{\delta}{d}$ (see [39], [36, Lemma 8], [1]). If $\theta > 0$, then Assumption C holds; condition 5 of this assumption follows from Lemma 2 in [42]. If $\theta = 0$, then Assumption D holds. In both cases we have $\kappa = \frac{d}{q} - \beta_v$, $\alpha_u = \alpha_g$, $\alpha_w = \alpha_v$. In the case $\theta = 0$ we have $\lambda_u = \lambda_g$, $\lambda_w = \lambda_v$. By Theorem 1, we obtain the desired upper estimates of entropy numbers. □

5 Estimates for entropy numbers of two-weighted summation operators on a tree

Applying Remark 2, we obtain estimates for entropy numbers of weighted summation operators on trees.

Let $\mathcal{G}$ be a graph. Given a function $f : \mathbf{V}(\mathcal{G}) \to \mathbb{R}$, we set

$$\|f\|_{l_p(\mathcal{G})} = \left(\sum_{\xi\in\mathbf{V}(\mathcal{G})} |f(\xi)|^p\right)^{1/p} \quad \text{for} \quad 1 \leqslant p < \infty, \quad \|f\|_{l_\infty(\mathcal{G})} = \sup_{\xi\in\mathbf{V}(\mathcal{G})} |f(\xi)|. \tag{155}$$

Denote by $l_p(\mathcal{G})$ the space of functions $f : \mathbf{V}(\mathcal{G}) \to \mathbb{R}$ with finite norm $\|f\|_{l_p(\mathcal{G})}$.

Let $\mathcal{A}$ be a tree, $\mathbf{V}(\mathcal{A}) = \{\eta_{j,i} : \ j \in \mathbb{Z}_+, \ \ i \in I_j\}$, let $\eta_{0,1}$ be the minimal vertex of $\mathcal{A}$, and let $\mathbf{V}^{\mathcal{A}}_j(\eta_{0,1}) = \{\eta_{j,i}\}_{i\in I_j}$ for any $j \in \mathbb{Z}_+$. Suppose that for some $c_* \geqslant 1$, $m_* \in \mathbb{N}$

$$c_*^{-1}\frac{h(2^{-m_*j})}{h(2^{-m_*(j+l)})} \leqslant \mathbf{V}^{\mathcal{A}}_l(\eta_{j,i}) \leqslant c_*\frac{h(2^{-m_*j})}{h(2^{-m_*(j+l)})}, \quad j, \ l \in \mathbb{Z}_+,$$

where the function $h \in \mathbb{H}$ is defined by (148) near zero. Let $u, \, w : \mathbf{V}(\mathcal{A}) \to (0, \, \infty)$, $u(\eta_{j,i}) = u_j$, $w(\eta_{j,i}) = w_j$, $j \in \mathbb{Z}_+$, $i \in I_j$,

$$u_j = 2^{\kappa m_* j}(m_*j+1)^{-\alpha_u}, \quad w_j = 2^{-\kappa m_* j}(m_*j+1)^{-\alpha_w}. \tag{156}$$

In addition, we suppose that

$$\kappa > \frac{\theta}{q} \quad \text{or} \quad \kappa = \frac{\theta}{q}, \quad \alpha_w > \frac{1-\gamma}{q}. \tag{157}$$

We set $\mathfrak{Z} = (p, \, q, \, u, \, w, \, h, \, m_*, \, c_*)$.

Denote $\tilde\alpha = \alpha_u + \alpha_w$ for $\kappa > \frac{\theta}{q}$, $\tilde\alpha = \alpha_u + \alpha_w - \frac{1}{q}$ for $\kappa = \frac{\theta}{q}$.

In [37] estimates for $e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \to l_q(\mathcal{A}))$ were obtained in the case $\tilde\alpha \ne \frac{1}{p} - \frac{1}{q}$. The case $\tilde\alpha = \frac{1}{p} - \frac{1}{q}$, $q = \infty$ was considered by Lifshits and Linde [25, 27]. To this end estimates for entropy numbers of the dual operator $S^*_{u,w,\mathcal{A}} : l_1(\mathcal{A}) \to l_{p'}(\mathcal{A})$ were obtained and the result of the paper [4] was applied.

Theorem 4. *Let $\theta > 0$, $1 < p < q < \infty$, $\tilde\alpha = \frac{1}{p} - \frac{1}{q}$. Then*

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \to l_q(\mathcal{A})) \underset{\mathfrak{Z}}{\asymp} n^{\frac{1}{q}-\frac{1}{p}}.$$

Now we suppose that $\theta = 0$, $\kappa > 0$, and instead of (156) the following condition holds:

$$u_j = 2^{\kappa m_* j}(m_*j+1)^{\alpha}[\log(m_*j+1)]^{-\lambda_u}, \quad w_j = 2^{-\kappa m_* j}(m_*j+1)^{-\alpha}[\log(m_*j+1)]^{-\lambda_w} \tag{158}$$

with $\alpha \in \mathbb{R}$.

Theorem 5. *Let $\theta = 0$, $\gamma \leqslant 0$, $1 < p < q < \infty$, $\kappa > 0$, $\lambda_u + \lambda_w = \frac{1}{p} - \frac{1}{q}$. Then*

$$e_n(S_{u,w,\mathcal{A}} : l_p(\mathcal{A}) \to l_q(\mathcal{A})) \underset{\mathfrak{Z}}{\asymp} n^{\frac{1}{q}-\frac{1}{p}}.$$

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